

# FITTING FORMULAS FOR HARMONIC LASING IN FEL AMPLIFIERS

E.A. Schneidmiller, M.V. Yurkov, DESY, Hamburg, Germany

## Abstract

One of the most important subjects of the high-gain FEL engineering is the calculation of the gain length, and fitting formulas are frequently used for this purpose. Here we refer to Ming Xie fitting formulas [1] and fitting formulas for optimized FEL written down in an explicit form in terms of the electron beam and undulator parameters [2]. In this paper we perform generalization of these fitting formulas to the case of harmonic lasing.

## INTRODUCTION

In order to calculate FEL gain length (and, therefore, saturation length) one has to solve an eigenvalue equation. Eigenvalue equation for harmonic lasing was derived in the framework of one-dimensional (1D) model in [3, 4], and a thorough 1D analysis can be found in [5]. Usually, more realistic 3D model is required to make conclusions on a possibility of practical realization of some option. Three-dimensional analysis was done in [6], where an eigenvalue equation was derived based on an approach developed in [1] for the fundamental frequency. However, this eigenvalue equation is rather complicated and can be solved only numerically. One can correctly calculate the gain length for a specific set of parameters, but it is very difficult to trace general dependencies and perform analysis of the parameter space.

In this paper we perform a parametrization of the solution of the eigenvalue equation for lasing at odd harmonics [6], and present explicit (although approximate) expressions for FEL gain length, optimal beta-function, and saturation length taking into account emittance, betatron motion, diffraction of radiation, energy spread and its growth along the undulator length due to quantum fluctuations of the undulator radiation. Considering 3rd harmonic lasing as a practical example, we come to the conclusion that it is much more robust than usually thought, and can be widely used at the present level of accelerator and FEL technology. We surprisingly find out that in many cases the 3D model of harmonic lasing gives more optimistic results than the 1D model. For instance, one of the results of our studies is that in a part of the parameter space, corresponding to the operating range of soft X-ray beamlines of X-ray FEL facilities, harmonics can grow faster than the fundamental mode.

## EIGENVALUE EQUATION

In Ref. [1] the eigenvalue equation for a high-gain FEL was derived that includes such important effects as diffraction of radiation, betatron motion of particles and longitudinal velocity spread due to emittance, energy spread in the

electron beam, frequency detuning. The eigenvalue equation is an integral equation which can be evaluated numerically for any particular parameter set with a desirable accuracy. The generalization of this eigenvalue equation to the case of harmonic lasing was done in [6]. Here we present the latter result for the growth rate of  $TEM_{nm}$  mode in a dimensionless form accepted in [7]:

$$\begin{aligned} \bar{\Phi}_{nm}(p) = & -\frac{h^2 A_{JJh}^2}{A_{JJ1}^2 (2i h B \hat{\Lambda} - p^2)} \int_0^\infty dp' p' \bar{\Phi}_{nm}(p') \\ & \times \int_0^\infty \frac{\zeta d\zeta}{(1 - i h B \hat{k}_\beta^2 \zeta/2)^2} \exp \left[ -\frac{h^2 \hat{\Lambda}_T^2 \zeta^2}{2} - (\hat{\Lambda} + i \hat{C}) \zeta \right] \\ & \times \exp \left[ -\frac{p^2 + p'^2}{4(1 - i h B \hat{k}_\beta^2 \zeta/2)} \right] I_n \left[ \frac{pp' \cos(\hat{k}_\beta \zeta)}{2(1 - i h B \hat{k}_\beta^2 \zeta/2)} \right] \end{aligned} \quad (1)$$

where  $h = 1, 3, 5, \dots$  is harmonic number,  $I_n$  is the modified Bessel function of the first kind. The normalized growth rate  $\hat{\Lambda} = \Lambda/\Gamma$  has to be found from numerical solution of the integral equation. The following notations are used here:  $\hat{r} = r/(\sigma\sqrt{2})$ ,  $B = 2\sigma^2\Gamma\omega_1/c$  is the diffraction parameter,  $\omega_1$  is the fundamental frequency,  $\sigma = \sqrt{\epsilon}\beta$  is the transverse rms size of the matched Gaussian beam, emittance  $\epsilon$  is simply given by  $\epsilon = \epsilon_n/\gamma$  with  $\epsilon_n$  being normalized rms emittance,  $\hat{k}_\beta = k_\beta/\Gamma$  is the betatron motion parameter,  $k_\beta = 1/\beta$  is the betatron wavenumber,  $\beta$  is the beta-function,  $\hat{\Lambda}_T^2 = \sigma_\gamma^2/(\bar{\rho}\gamma)^2$  is the energy spread parameter,  $\hat{C} = [k_w - \omega_h/(2hc\gamma_z^2)]/\Gamma$  is the detuning parameter,  $\omega_h \simeq h\omega_1$ ,  $\Gamma = \left[ A_{JJ1}^2 I \omega_1^2 \theta_s^2 (I_A c^2 \gamma_z^2 \gamma)^{-1} \right]^{1/2}$  is the gain factor,  $\bar{\rho} = c\gamma_z^2\Gamma/\omega_1$  is the efficiency parameter,  $\theta_s = K/\gamma$ ,  $K$  is the rms undulator parameter,  $\gamma$  is relativistic factor,  $\gamma_z^{-2} = \gamma^{-2} + \theta_s^2$ ,  $k_w$  is the undulator wavenumber,  $I$  is the beam current,  $I_A = 17$  kA is the Alfvén current,  $A_{JJh} = J_{(h-1)/2}(hK^2/2(1+K^2)) - J_{(h+1)/2}(hK^2/2(1+K^2))$ . The coupling factors for the 1st, 3rd, and 5th harmonics are shown in Fig. 1. When the rms undulator parameter  $K$  is large, the coupling factors are  $A_{JJ1} \simeq 0.696$ ,  $A_{JJ3} \simeq 0.326$ ,  $A_{JJ5} \simeq 0.230$ . Asymptotically for large  $h$  we have  $A_{JJh} \simeq 0.652 h^{-2/3}$ . Note that the scaling factors ( $\Gamma$ ,  $\bar{\rho}$ ) reflect the growth rate of the fundamental harmonic. The efficiency parameter  $\bar{\rho}$  is related to the corresponding parameter  $\rho$  [8] of the one-dimensional model as follows:  $\bar{\rho} = \rho B^{1/3}$ .

One can observe that the equation (1) can be rewritten such that it looks the same for all harmonics:

$$\begin{aligned} \bar{\Phi}_{nm}(p) = & -\frac{1}{2i\tilde{B}\tilde{\Lambda} - p^2} \int_0^\infty dp' p' \bar{\Phi}_{nm}(p') \\ & \times \int_0^\infty dx \frac{x}{(1 - i\tilde{B}\tilde{k}_\beta x/2)^2} \exp\left[-\frac{\tilde{\Lambda}_T^2 x^2}{2} - (\tilde{\Lambda} + i\tilde{C})x\right] \\ & \times \exp\left[-\frac{p^2 + p'^2}{4(1 - i\tilde{B}\tilde{k}_\beta x/2)}\right] I_n\left[\frac{pp' \cos(\tilde{k}_\beta x)}{2(1 - i\tilde{B}\tilde{k}_\beta x/2)}\right], \quad (2) \end{aligned}$$

with the following scaling factors:  $\tilde{\Gamma} = \left[A_{JJh}^2 I\omega_h^2 \theta_s^2 (I_A c^2 \gamma_z^2 \gamma)^{-1}\right]^{1/2}$  and  $\tilde{\rho} = c\gamma_z^2 \tilde{\Gamma} / \omega_h$ . Note that the gain parameter can be rewritten as

$$\tilde{\Gamma} = \left(\frac{A_{JJh}^2 I\omega_h^2 K^2 (1 + K^2)}{I_A c^2 \gamma^5}\right)^{1/2} \quad (3)$$

The new scaled parameters are now written as follows:  $\tilde{\Lambda}_T^2 = \sigma_\gamma^2 / (\tilde{\rho}\gamma)^2$  is the energy spread parameter,  $\tilde{k}_\beta = k_\beta / \tilde{\Gamma}$  is the betatron motion parameter,  $\tilde{C} = [k_w - \omega_h / (2hc\gamma_z^2)] / \tilde{\Gamma}$  is the detuning parameter, and

$$\tilde{B} = 2\sigma^2 \tilde{\Gamma} \omega_h / c \quad (4)$$

is the diffraction parameter.

In this paper we concentrate on the case when beta-function is optimized for the highest FEL gain. Since diffraction parameter depends on beta-function, it is more convenient to go over to the normalized parameters other than those introduced above. Indeed, the diffraction parameter can be rewritten as  $\tilde{B} = 2\tilde{\epsilon} / \tilde{k}_\beta$ , where  $\tilde{\epsilon} = 2\pi\epsilon / \lambda_h$  and  $\lambda_h = 2\pi c / \omega_h$ . Then we can go from parameters  $(\tilde{B}, \tilde{k}_\beta)$  to  $(\tilde{\epsilon}, \tilde{k}_\beta)$ , and the Eq. (2) becomes

$$\begin{aligned} \bar{\Phi}_{nm}(p) = & -\frac{1}{4i\tilde{\epsilon}\tilde{\Lambda}/\tilde{k}_\beta - p^2} \int_0^\infty dp' p' \bar{\Phi}_{nm}(p') \\ & \times \int_0^\infty dx \frac{x}{(1 - i\tilde{\epsilon}\tilde{k}_\beta x)^2} \exp\left[-\frac{\tilde{\Lambda}_T^2 x^2}{2} - (\tilde{\Lambda} + i\tilde{C})x\right] \\ & \times \exp\left[-\frac{p^2 + p'^2}{4(1 - i\tilde{\epsilon}\tilde{k}_\beta x)}\right] I_n\left[\frac{pp' \cos(\tilde{k}_\beta x)}{2(1 - i\tilde{\epsilon}\tilde{k}_\beta x)}\right]. \quad (5) \end{aligned}$$

Our goal is to find the reduced growth rate (the real part of the eigenvalue)  $Re\tilde{\Lambda} = Re\Lambda / \tilde{\Gamma}$  of the transverse mode  $TEM_{00}$  when an FEL lases at  $h$ -th harmonic. The field gain length of this mode is then simply  $L_g = 1/Re\Lambda$ . In the case of a SASE FEL the detuning parameter falls out of the parameters of the problem since the lasing always takes place at the optimal detuning. Thus, when solving the eigenvalue equation, we should always find the eigenvalue at the optimal detuning. Let us also assume at the first step that the energy spread parameter is negligibly small (denoting the gain length for this case as  $L_{g0}$ ), so that its influence on FEL operation can be neglected. In this case

the reduced growth rate  $Re\Lambda$  depends only on two dimensionless parameters:  $\tilde{\epsilon}$  and  $\tilde{k}_\beta$ . If in addition one optimizes beta-function, then the reduced growth rate is the function of the only parameter, scaled emittance:  $Re\tilde{\Lambda} = f(\tilde{\epsilon})$ . Correspondingly, the field gain length can be written as follows:

$$L_{g0} = [\tilde{\Gamma} f(\tilde{\epsilon})]^{-1} \quad (6)$$

Numerical solution of the eigenvalue equation (5) is time-consuming, so we used an approximate solution [7] which agrees very well (to better than 1% in the whole parameter space) with the solution of Eq. (5). In the most interesting parameter range,  $1 < \tilde{\epsilon} < 5$ , we have found [2] that the function  $f(\tilde{\epsilon})$  is well approximated as  $f(\tilde{\epsilon}) \propto \tilde{\epsilon}^{-5/6}$ , so that the gain length in the case of negligible energy spread and optimal beta-function is

$$L_{g0} \simeq a_1 \tilde{\Gamma}^{-1} \tilde{\epsilon}^{5/6}, \quad (7)$$

where  $a_1$  is the fitting coefficient. Now we would like to include the effects of the energy spread. For that we present the growth rate as  $L_g = L_{g0}(1 + \delta)$ , where  $\delta$  depends on the energy spread. Again, for the optimal beta-function, we found that the fit  $\delta \propto \tilde{\Lambda}_T^2 \tilde{\epsilon}^{5/4}$  works very well in the wide range of values of the energy spread parameter. Thus, the field gain length for the optimal beta-function can be written as follows:

$$L_g \simeq a_1 \tilde{\Gamma}^{-1} \tilde{\epsilon}^{5/6} (1 + a_2 \tilde{\Lambda}_T^2 \tilde{\epsilon}^{5/4}). \quad (8)$$

Optimizing fitting coefficients  $a_1$  and  $a_2$  in the range of parameters, specified in (13), (14), we obtain the Eqs. (10)-(12). In a similar way we obtained the expression (15) for the optimal beta-function. In particular, in the case of negligibly small energy spread we used the following approximation:  $(\tilde{k}_\beta)_{opt} \propto \tilde{\epsilon}^{-3/2}$ .

## GAIN LENGTH OF HARMONIC LASING

The results of this Section are generalizations of the results of Ref. [2] for the fundamental frequency to the case of harmonic lasing. Let us consider an axisymmetric electron beam with a current  $I$ , and a Gaussian distribution in transverse phase space and in energy. The resonance condition for the fundamental wavelength is written as:

$$\lambda_1 = \frac{\lambda_w (1 + K^2)}{2\gamma^2}. \quad (9)$$

More generally, lasing in a planar undulator can be achieved at the odd harmonics defined by the condition

$$\lambda_h = \frac{\lambda_1}{h}, \quad h = 1, 3, 5, \dots$$

Here  $\lambda_w$  is the undulator period,  $\gamma$  is relativistic factor,  $K = 0.934 \times \lambda_w [\text{cm}] \times B_{\text{rms}} [\text{T}]$  is the rms undulator parameter, and  $B_{\text{rms}}$  is the rms undulator field.

In what follows we assume that the harmonic with a number  $h$  lases to saturation, while lasing at harmonics

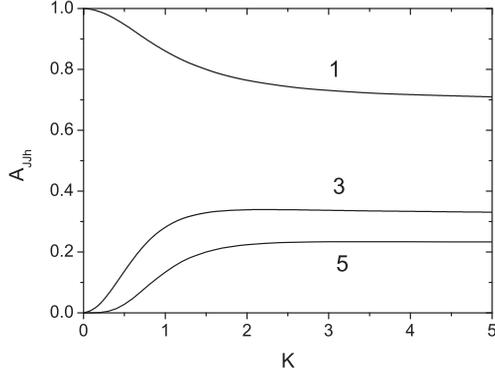


Figure 1: Coupling factors for the 1st, 3rd, and 5th harmonics (denoted with 1, 3, and 5, correspondingly) versus rms undulator parameter.

with lower numbers and at the fundamental wavelength is suppressed with the help of phase shifters or by other means (see [9] for details). We also assume that the beta-function is optimized so that the FEL gain length at a considered harmonic achieves the minimum for given wavelength, beam and undulator parameters. Under this condition the solution of the eigenvalue equation for the field gain length of the  $TEM_{00}$  mode can be approximated according to (8):

$$L_g \simeq L_{g0} (1 + \delta), \quad (10)$$

where

$$L_{g0} = 1.67 \left( \frac{I_A}{I} \right)^{1/2} \frac{(\epsilon_n \lambda_w)^{5/6}}{\lambda_h^{2/3}} \frac{(1 + K^2)^{1/3}}{h^{5/6} K A_{JJh}}, \quad (11)$$

and

$$\delta = 131 \frac{I_A}{I} \frac{\epsilon_n^{5/4}}{\lambda_h^{1/8} \lambda_w^{9/8}} \frac{h^{9/8} \sigma_\gamma^2}{(K A_{JJh})^2 (1 + K^2)^{1/8}}. \quad (12)$$

Here  $\epsilon_n = \gamma \epsilon$  is the rms normalized emittance,  $\sigma_\gamma = \sigma_\epsilon / mc^2$  is the rms energy spread in units of the rest electron energy. Also note that all the formulas of this Section are valid in the case of helical undulator and the fundamental wavelength ( $h = 1$ ), in this case the coupling factor is equal to 1 [2].

The formulas (10)-(12) provide an accuracy better than 5 % in the range of parameters

$$1 < \frac{2\pi\epsilon}{\lambda_h} < 5, \quad (13)$$

$$\delta < 2.5 \left\{ 1 - \exp \left[ -\frac{1}{2} \left( \frac{2\pi\epsilon}{\lambda_h} \right)^2 \right] \right\} \quad (14)$$

In fact, the formulas (10)-(12) can also be used well beyond this range, but the above mentioned accuracy is not guaranteed.

We also present here an approximate expression for the optimal beta-function (an accuracy is about 10 % in the above mentioned parameter range):

$$\beta_{\text{opt}} \simeq 11.2 \left( \frac{I_A}{I} \right)^{1/2} \frac{\epsilon_n^{3/2} \lambda_w^{1/2}}{\lambda_h h^{1/2} K A_{JJh}} (1 + 8\delta)^{-1/3} \quad (15)$$

To estimate the saturation length, one can use the result from Ref. [10], generalized to the case of harmonic lasing:

$$L_{\text{sat}} \simeq 0.6 L_g \ln \left( h N_{\lambda_h} \frac{L_g}{\lambda_w} \right). \quad (16)$$

Here  $N_{\lambda_h}$  is a number of electrons per wavelength of the considered harmonic. For operating VUV and X-ray SASE FELs one typically has  $L_{\text{sat}} \simeq (10 \pm 1) \times L_g$ .

Energy spread in the electron beam grows along the undulator length due to the quantum diffusion [11, 12]. In this case an effective parameter  $\delta$  can be introduced in order to describe an increase in saturation length due to this effect, see [9]. Let us also note that all the above presented results are reduced to those of Ref. [2] for the case of the first harmonic ( $h = 1$ ). All these results were obtained under the assumption that beta-function is optimal (i.e. it is given by Eq. (15)). However, for technical reasons this is not always the case in real machines, and it could often be that  $\beta > \beta_{\text{opt}}$ . In such a case the gain length can be approximated as follows:

$$L_g(\beta) \simeq L_g(\beta_{\text{opt}}) \left[ 1 + \frac{(\beta - \beta_{\text{opt}})^2 (1 + 8\delta)}{4\beta_{\text{opt}}^2} \right]^{1/6}. \quad (17)$$

Finally, let us note that widely used Ming Xie formulas [13, 14] can be easily generalized to the case of harmonic lasing. Comparing two approaches to parametrization of FEL gain length, we have found that they agree reasonably well, also for non-optimal beta-functions and well beyond the range given by Eq. (13).

## GENERALIZATION OF MING XIE FORMULAS

In Refs. [13, 14] the fitting formulas were presented that approximate FEL power gain length,  $L_g$ . Note that in our parametrization we use the same notation for the field gain length which is twice longer. The power gain length of the fundamental harmonic was expressed in [13, 14] as follows:

$$\frac{L_{1d}}{L_g} = \frac{1}{1 + \Lambda(\eta_d, \eta_\epsilon, \eta_\gamma)}, \quad (18)$$

where  $L_{1d}$  is the 1D gain length for the cold beam, and  $\Lambda$  depends on the three dimensionless parameters:  $\eta_d$ ,  $\eta_\epsilon$ , and  $\eta_\gamma$ . This dependence can be found in [13, 14], it was

obtained by fitting the solution of the eigenvalue equation with the help of 19 fitting coefficients.

We can generalize these results for calculation of power gain length  $L_g^{(h)}$  of harmonic lasing in a simple way. Eq. (18) can be generalized as

$$\frac{L_{1d}^{(h)}}{L_g^{(h)}} = \frac{1}{1 + \Lambda(\eta_d^{(h)}, \eta_\epsilon^{(h)}, \eta_\gamma^{(h)})}. \quad (19)$$

The 1D gain length of harmonics can be calculated as

$$L_{1d}^{(h)} = \left( \frac{A_{JJ1}^2}{hA_{JJh}^2} \right)^{1/3} L_{1d},$$

and the function  $\Lambda$  now depends on the three generalized parameters:

$$\eta_d^{(h)} = \left( \frac{A_{JJ1}^2}{hA_{JJh}^2} \right)^{1/3} \frac{\eta_d}{h}, \quad \eta_\epsilon^{(h)} = \left( \frac{A_{JJ1}^2}{hA_{JJh}^2} \right)^{1/3} h\eta_\epsilon,$$

$$\eta_\gamma^{(h)} = \left( \frac{A_{JJ1}^2}{hA_{JJh}^2} \right)^{1/3} h\eta_\gamma.$$

## COMPARISON OF THE TWO APPROACHES

We present a comparison for the case of LCLS. The main parameters are as follows [15]: undulator period is 3 cm, rms undulator parameter is 2.475, peak current of the electron bunch is 3 kA, normalized emittance is 0.4 mm mrad, slice energy spread is 1.4 MeV. Beta function scales with electron energy as  $\beta[m] = 30 \frac{E[\text{GeV}]}{13.6}$ . In Fig. 2 we present the power gain length versus wavelength for lasing at the fundamental and at the third harmonic, calculated with our formulas and with generalized Ming Xie formulas. One can notice a good agreement of two different parametrizations of the FEL gain length. It is also worth noticing that in the range of wavelengths 1.5 - 5 Å the third harmonic gain length is slightly smaller than that of the fundamental (achieved at a larger electron energy).

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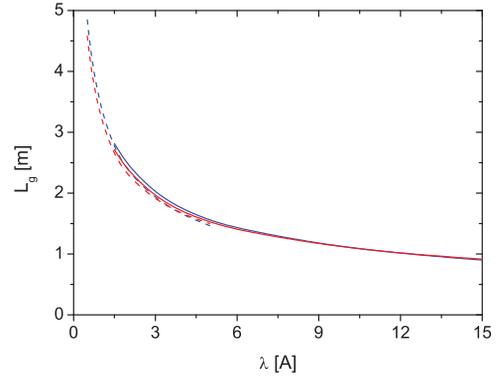


Figure 2: Power gain length of the fundamental (solid) and of the 3rd harmonic (dash) versus wavelength for LCLS. Shown in blue are results of calculation with our formulas, in red - with generalized Ming Xie formulas.

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