NONLINEAR EFFECTS IN FEL THEORY AND THEIR ROLE IN COHERENT ELECTRON COOLING*

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Abstract

The novel cooling technique, the coherent electron cooling [1] relies on the amplification of the interaction between hadrons and electrons by an FEL. The linearity of the amplification process is essential for operation of such cooler. In this paper we propose a theoretical method of taking into account nonlinear effects in computation of evolution of charge perturbation in an FEL. This will allow to explore the limits of the FEL gain with special attention to the smearing of the phase caused by nonlinear and saturation effects.

INTRODUCTION

The coherent electron cooling (CeC) is a realization of the stochastic cooling where an electron beam copropagating with a hadron beam being cooled serves as a pick-up and a kicker being amplified by an FEL on its way, the schematic layout of the device is depicted in Fig. 1. Pick-up or modulator stores the information about the hadron beam as a perturbation of a charge density in the electron beam, then this perturbation is amplified by an FEL, and then goes to the kicker where its field accelerates slow moving hadrons and decelerates the fast ones. The detailed theoretical investigation of all these steps is required to build a working device. Now the modulator section is described in an infinite beam approximation in [2] and the study for the finite realistic beam is started in [3]. In the CeC an FEL is used in a nonstandard way, i.e. as an amplifier of the electron density perturbation. This facet of the FEL theory is not developed enough and we fill this gap in this article. With the methods presented we plan to analyze possible limitations of the FEL gain in the CeC device by nonlinear effects and saturation. In the next two sections we discuss 1D model for a modulator and the simplest possible initial conditions, i.e. electron density perturbation coming from the modulator. We considered 1D model and cos-like condition. The rest of the paper is devoted to possible ways to compute an evolution of these perturbations in an FEL.

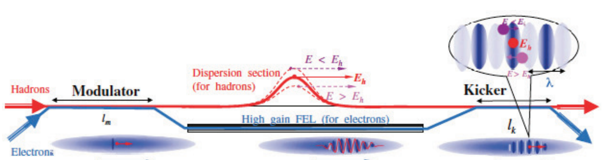

Figure 1: The scheme of the coherent electron cooler.

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1D MODULATOR

In 1D case the problem of a shielding of a hadron moving along the trajectory $y(t) = x_0 + tv_0$ in an electron beam is described by the following Maxwell-Vlasov system:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = \frac{e}{m_0\gamma}\frac{\partial U}{\partial x}\cdot\frac{\partial f_0}{\partial v}, \tag{1}$$

$$\frac{\partial^2 U(x,t)}{\partial x^2} = -\frac{e}{\epsilon_0}\left(n_1(x,t) - Z\delta(x - y(t))\right), \tag{2}$$

the Maxwell equation can be solved via Fourier transform:

$$k^2\tilde{U}(k,t) = \frac{e}{\epsilon_0}\left(\tilde{n}_1(x,t) - Z\int\delta(x - y(t))\,\mathrm{e}^{-ikx}dx\right). \tag{3}$$

Using this solution the equation (1) can be transformed to

$$\tilde{N}_1(k,s) = -\frac{e}{\epsilon_0}\int_0^\infty \mathrm{e}^{-ts}t\int f_0(v)\,\mathrm{e}^{-ikvt}dvdt\times$$
$$\times\left(\tilde{N}_1(k,s) - Z\int_0^\infty \mathrm{e}^{-iky(t)-ts}dt\right), \tag{4}$$

where $\tilde{N}_1(k,s)$ is a Laplace-Fourier image of $n_1(x,t) \equiv \int f_1 dv$. Assuming the cold electron beam we have

$$\tilde{N}_1(k,s) = Z\rho\frac{\mathrm{e}^{-ikx_0}}{(s + ikv_0)\left((s + ikv_c)^2 + \frac{e\rho}{\epsilon_0}\right)}. \tag{5}$$

The inverse Laplace and Fourier transforms of this expression can be computed by Mathematica analytically, the expression is pretty bulky. It appears to be complex. Looking back to initial equations and assuming complex f_1 we see that equation with $\mathrm{Im} f_1$ corresponds to equation without external charge, while equation with $\mathrm{Re} f_1$ is the equation with it. So as a solution we take

$$n_1(x,t) = \mathrm{Re}\mathcal{F}^{-1}\mathcal{L}^{-1}\left\{\frac{Z\rho\mathrm{e}^{-ikx_0}}{(s + ikv_0)\left((s + ikv_c)^2 + \frac{e\rho}{\epsilon_0}\right)}\right\}. \tag{6}$$

INITIAL CONDITIONS

We change variables in (6) to the ones widely used in FEL theory [4], namely we take $z \equiv x$ as a new independent variable and $\theta = k_w z + \omega\left(\frac{z}{c} - r\right)$ and get:

$$n_1(\theta,z) = \mathrm{Re}\int\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{-iZ\rho\mathrm{e}^{-ikx_0+ikz+\left(\frac{z}{c}-\frac{\theta-k_w z}{\omega}\right)s}}{(s + ikv_0)\left((s + ikv_c)^2 + \frac{e\rho}{\epsilon_0}\right)}dkds, \tag{7}$$

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and

$$n_1(\nu,z)= = \frac{Z\rho}{2\pi}\mathrm{Re}\int \frac{\mathrm{e}^{-ikx_0+ikz-i\frac{\omega}{\nu}z\left(\frac{1}{c}+\frac{k_w}{\omega}\right)}}{\left(\frac{\omega}{\nu}-kv_0\right)\left(\frac{e\rho}{\epsilon_0}-\left(kv_c-\frac{\omega}{\nu}\right)^2\right)}dk, \quad (8)$$

so

$$F^{(1)}(\nu,0)=\frac{Z\rho}{2\pi}\mathrm{Re}\int \frac{\mathrm{e}^{-ikx_0}}{\left(\frac{\omega}{\nu}-kv_0\right)\left(\frac{e\rho}{\epsilon_0}-\left(kv_c-\frac{\omega}{\nu}\right)^2\right)}dk. \quad (9)$$

We also consider function of the form $\cos(\omega_0 t+\varphi)$, changing variables to θ, z and setting $z=0$ we have the following initial condition:

$$\tilde{F}^{(1)}(\theta,\eta,0)=F_0^{(1)}\cos(\alpha\theta+\varphi),\quad \alpha>0 \quad (10)$$

$$F^{(1)}(\nu,0)=\pi F_0^{(1)}\left(\mathrm{e}^{-i\varphi}\delta(\nu-\alpha)+\mathrm{e}^{i\varphi}\delta(\nu+\alpha)\right). \quad (11)$$

LINEAR CASE

Generally a 1D FEL is described by the following nonlinear system of Maxwell-Vlasov equations [4]

$$\left(\frac{\partial}{\partial z}-i\Delta\nu k_u\right)A(\nu,z)= \quad (12)$$

$$=-\frac{K[JJ]jZ_0}{4\gamma_0\omega_1\sqrt{2\pi}}\int\int \mathrm{e}^{i\nu\theta}\tilde{F}(\theta,\eta,z)\,d\theta d\eta, \quad (13)$$

$$\left(\frac{\partial}{\partial z}+2k_u\eta\frac{\partial}{\partial\theta}\right)\tilde{F}(\theta,\eta,z)+\frac{eK[JJ]}{2\gamma_0^2mc^2\sqrt{2\pi}}\times \times\int A(\nu,z)\,\mathrm{e}^{-i\nu\theta}d\nu\frac{\partial}{\partial\eta}\tilde{F}(\theta,\eta,z)=0. \quad (14)$$

The Vlasov equation can be solved via method of the orbits with the orbit $\theta^{(0)}(z_1)=\theta+2k_u\eta(z_1-z)$, sometimes we use notation $\phi\equiv 2k_u\eta$, for the general nonlinear case we have:

$$\tilde{F}(\theta,\eta,z)=\tilde{F}\left(\theta^{(0)}(0),\eta,0\right)-\frac{eK[JJ]}{2\gamma_0^2mc^2\sqrt{2\pi}}\times \times\int_0^z\int A(\nu,z_1)\,\mathrm{e}^{-i\nu\theta^{(0)}(z_1)}d\nu\frac{\partial}{\partial\eta}\tilde{F}\left(\theta^{(0)}(z_1),\eta,z_1\right)dz_1, \quad (15)$$

linear approximation of this equation can be obtained by setting $\tilde{F}(\theta,\eta,z)=\tilde{F}^{(0)}(\eta)=n_0\delta(\eta-\eta_0)$ in the integral. Solutions of the linearized system we will denote $\tilde{F}^{(1)}(\theta,\eta,z)$ and $A^{(1)}(\nu,z)$. Plugging the solution for the linearized case into the Maxwell equation we have:

$$\left(\frac{\partial}{\partial z}-i\Delta\nu k_u\right)A^{(1)}(\nu,z)=-\frac{K[JJ]jZ_0}{4\gamma_0\omega_1\sqrt{2\pi}}\times \times\int\int \mathrm{e}^{i\nu\theta}\tilde{F}^{(1)}\left(\theta^{(0)}(0),\eta,0\right)d\theta d\eta+ +2\pi\rho_1^3\int_0^z\int A^{(1)}(\nu,z_1)\,\mathrm{e}^{-i\nu 2k_u\eta(z_1-z)}\frac{\partial}{\partial\eta}\tilde{F}^{(0)}(\eta)\,d\eta dz_1, \quad (16)$$

where $\rho_1=\left(eK^2[JJ]^2jZ_0/\left(8\gamma_0^3mc^2\omega_1 2\pi\right)\right)^{\frac{1}{3}}$, the first term in the right hand side is a contribution of initial perturbation. This equation can be solved via Laplace transform giving for the Laplace image

$$\bar{A}^{(1)}(\nu,s)=\frac{-A^{(1)}(\nu,0)+d_1\int\frac{F^{(1)}(\nu,\eta,0)}{s+2i\nu k_u\eta}d\eta}{-s+i\Delta\nu k_u-2\pi\rho_1^3\frac{4i\pi\nu k_u n_0}{(s+2i\nu k_u\eta_0)^2}}, \quad (17)$$

where $F^{(1)}(\nu,\eta,0)$ is known initial perturbation and $d_1=K[JJ]jZ_0/(4\gamma_0\omega_1)$. Plugging this expression into the solution of the Vlasov equation and doing some integrations we get

$$\tilde{F}^{(1)}(\theta,\eta,z)=\tilde{F}^{(1)}\left(\theta^{(0)}(0),0\right)\delta(\eta-\eta_1)-\frac{n_0\rho_1^3}{d_1}\frac{\partial}{\partial\eta}\delta(\eta-\eta_0)\times \times\sum_{j=1}^{3}\frac{\int\frac{1}{\nu^{\frac{2}{3}}}\left(-A^{(1)}(\nu,0)+\frac{d_1F^{(1)}(\nu,0)}{s_j+2i\nu k_u\eta_1}\right)\frac{\mathrm{e}^{-i\nu\theta}\left(\mathrm{e}^{s_jz}-\mathrm{e}^{2i\eta k_u\nu z}\right)}{s_j-2i\eta k_u\nu}d\nu}{(2\pi)^{-1/2}\prod_{k=1,\,j\neq k}^{3}\left(s_j/\nu^{\frac{1}{3}}-s_k/\nu^{\frac{1}{3}}\right)}, \quad (18)$$

where s_j are the poles of the expression (17), for simplicity we take $\Delta\nu=0$ and $\eta_0=0$, in this case $\prod_{k=1,\,j\neq k}^{3}\left(s_j/\nu^{\frac{1}{3}}-s_k/\nu^{\frac{1}{3}}\right)$ doesn't depend on ν. This expression has to be integrated over energies η, this is straightforward because of the delta-functions, integration over ν has to be computed numerically for the initial condition (9) and can be done analytically for (11):

$$\tilde{F}^{(1)}(\theta,\eta,z)=\tilde{F}_0^{(1)}\cos\left(\alpha\theta^{(0)}(0)+\phi\right)\delta(\eta-\eta_1)- -\pi n_0\sqrt{2\pi}\rho_1^3\sum_{j=1}^{3}\frac{\sum_{\nu=\pm\alpha}\frac{1}{\nu^{\frac{2}{3}}}\frac{\mathrm{e}^{-\mathrm{sign}(\nu)i\varphi}}{s_j+2i\nu k_u\eta_1}\frac{\mathrm{e}^{-i\nu\theta}\left(\mathrm{e}^{s_jz}-\mathrm{e}^{2i\eta k_u\nu z}\right)}{s_j-2i\eta k_u\nu}}{\prod_{k=1,\,j\neq k}^{3}\left(s_j/\nu^{\frac{1}{3}}-s_k/\nu^{\frac{1}{3}}\right)}\times \times F_0^{(1)}\frac{\partial}{\partial\eta}\delta(\eta-\eta_0), \quad (19)$$

where $A^{(1)}(\nu,0)$ is assumed to be zero and η_1 is energy of initial mono-energetic perturbation. The electron density perturbation of the second order can be obtained in the similar way, plugging first order perturbation into the expression (15), but this leads to an equation which is not solvable by Laplace transform, at least without significant simplifications.

NONLINEAR CASE VIA EIGENFUNCTION EXPANSION

To treat the non-linear case we employ the Van-Kampen method of expansion over the eigenfunctions. The Maxwell equation and the Fourier transformed to ν-space Vlasov equation in the first order can be written in the matrix form:

$$\left(\frac{\partial}{\partial z}-i\mathcal{M}\right)\Phi=0, \quad (20)$$

where

$$\Phi=\begin{pmatrix}A(\nu,z)\\ F(\nu,\eta,z)\end{pmatrix},\ \mathcal{M}=\begin{pmatrix}\Delta\nu k_u & i\frac{K[JJ]jZ_0}{4\gamma_0\omega_1}\int d\eta\\ i\frac{eK[JJ]}{2\gamma_0^2mc^2}\frac{\partial}{\partial\eta}F^{(0)}(\eta) & -2k_u\eta\nu\end{pmatrix}, \quad (21)$$

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Introducing dot product

$$[\Phi_1, \Phi_2] = a_1(\nu)\, a_2(\nu) + \int f_1(\nu)\, f_2(\nu)\, d\eta, \quad (22)$$

and defining adjoint functions and operator via $\left[\mathcal{M}^\dagger \Phi_1^\dagger, \Phi_2\right] = \left[\Phi_1^\dagger, \mathcal{M}\Phi_2\right]$ we have for them

$$\Phi^\dagger = \left(A(\nu, z), \quad F(\nu, \eta, z)\right), \quad (23)$$

$$\mathcal{M}^\dagger = \begin{pmatrix} \Delta\nu k & i\frac{eK[JJ]}{2\gamma_0^2 mc^2} \int d\eta \frac{\partial}{\partial\eta} F^{(0)}(\eta) \\ i\frac{K[JJ]jZ_0}{4\gamma_0\omega_1} & -2k_u\eta\nu \end{pmatrix} \quad (24)$$

Looking for the solution in this form:

$$\Phi^{(1)} \equiv \begin{pmatrix} A^{(1)}(\nu, z) \\ F^{(1)}(\nu, \eta, z) \end{pmatrix} = \mathrm{e}^{-i\mu_n^{(1)} z} \begin{pmatrix} \mathcal{A}_n^{(1)} \\ \mathcal{F}_n^{(1)}(\eta) \end{pmatrix} \equiv \mathrm{e}^{-i\mu_n^{(1)} z}\Psi_n. \quad (25)$$

we have

$$\mathcal{F}_n^{(1)}(\eta) = -\frac{eK[JJ]}{2\gamma_0^2 mc^2} \left(\frac{\partial}{\partial\eta} F^{(0)}(\eta)\right) \mathcal{A}_n^{(1)} \frac{i}{\mu_n^{(1)} - 2k_u\eta\nu}, \quad (26)$$

with condition $\mathrm{Im}\left(2\nu k_u\eta - \mu_n^{(1)}\right) < 0$. Lets look at the Maxwell equation:

$$\left(-i\mu_n^{(1)} - i\Delta\nu k_u\right) \mathcal{A}_n^{(1)} = -\frac{K[JJ]jZ_0}{4\gamma_0\omega_1} \int \mathcal{F}_n^{(1)}(\eta)\, d\eta, \quad (27)$$

plugging the density we have

$$\left(-i\mu_n^{(1)} - i\Delta\nu k_u\right) \mathcal{A}_n^{(1)} = 2\pi\rho_1^3 \int \left(\frac{\partial}{\partial\eta} F^{(0)}(\eta)\right) \times$$
$$\times \mathcal{A}_n^{(1)} \int_{-\infty}^{0} \mathrm{e}^{i\left(2\nu k_u\eta - \mu_n^{(1)}\right)\tau} d\tau d\eta, \quad (28)$$

For the KV equilibrium distribution this equation reduces to:

$$\mu_n^{(1)} + \Delta\nu k_u = 4\pi k_u \nu \rho_1^3 n_0 \frac{1}{\left(\mu_n^{(1)} - 2k_u\eta_0\nu\right)^2}, \quad (29)$$

which is the same as the equation for the poles of (17). Similarly we can solve adjoint equation $\left(\mu^\dagger + \mathcal{M}^\dagger\right)\Psi_n^\dagger = 0$. We have $\mu^\dagger = \mu$, ${\mathcal{A}_n^{(1)}}^\dagger = \mathcal{A}_n^{(1)}$ and

$$\mathcal{F}^\dagger(\eta) = -i\frac{K[JJ]jZ_0}{4\gamma_0\omega_1} \frac{\mathcal{A}_n^{(1)}}{\mu_n^{(1)} - 2k_u\eta\nu} \quad (30)$$

If we have non-trivial initial conditions the solution is the following:

$$\Phi(\nu, z) = \sum_n \mathrm{e}^{-i\mu_n^{(1)} z} \frac{\left[\Psi_n^\dagger, \Phi(0)\right]}{\left[\Psi_n^\dagger, \Psi_n\right]} \Psi_n \equiv \sum_n c_n^{(1)}(\nu, z)\, \Psi_n, \quad (31)$$

where

$$\left[\Psi_n^\dagger, \Phi(0)\right] = A^{(0)}(\nu)\, \mathcal{A}_n^{(1)} + \int F^{(0)}(\nu, \eta, 0)\, \mathcal{F}_n^{(1)}(\eta)\, d\eta, \quad (32)$$

$$\left[\Psi_n^\dagger, \Psi_n\right] = {\mathcal{A}_n^{(1)}}^2 + \int {\mathcal{F}_n^{(1)}(\eta)}^\dagger \mathcal{F}_n^{(1)}(\eta)\, d\eta = \quad (33)$$

$$= \mathcal{A}_n^{(1)} + \frac{8\pi i\rho_1^3 n_0 k_u \nu \mathcal{A}_n^{(1)}}{\left(\mu_n^{(1)} - 2k_u\eta_0\nu\right)^3} \quad (34)$$

The initial distributions we consider doesn't depend on η, so for them

$$\left[\Psi_n^\dagger, \Phi(0)\right] = A^{(0)}(\nu)\, \mathcal{A}_n^{(1)} - 2ik_u \frac{n_0\nu\mathcal{A}_n^{(1)} F^{(0)}(\nu, 0)}{\left(\mu_n^{(1)} - 2\eta_0 k_u\nu\right)^2}. \quad (35)$$

This method doesn't give an equation for $\mathcal{A}_n^{(1)}$, it can be considered as a parameter or can be found from previous method. For the 3D case this method also works and gives an integral equation for the $\mathcal{A}_n^{(1)}(\vec{x}_\perp)$, where $\vec{x}_\perp$ is a transverse coordinate.

For the second order plugging the solution of the Vlasov equation into the Maxwell one we have:

$$\left(\frac{\partial}{\partial z} - i\Delta\nu k_u\right) A^{(2)}(\nu, z) = \rho_1^3 \int\int \mathrm{e}^{i\nu\theta} \int_0^z \int A^{(2)}(\nu_2, z_2) \times$$
$$\times\, \mathrm{e}^{-i\nu_2\theta^{(0)}(z_2)} d\nu_2 \frac{\partial}{\partial\eta} \tilde{F}^{(1)}\left(\theta^{(0)}(z_2), \eta, z_2\right) dz_2 d\theta d\eta. \quad (36)$$

And we have for the density:

$$\tilde{F}^{(1)}\left(\theta^{(0)}(z_2), \eta, z_2\right) =$$
$$= \sum_n \int c_n^{(1)}(\nu_1, z_2)\, \mathcal{F}_n^{(1)}(\eta)\, \mathrm{e}^{-i\nu_1\theta^{(0)}(z_2)} d\nu_1. \quad (37)$$

Here we also look for solution in such form:

$$\Phi^{(2)} = \begin{pmatrix} A^{(2)}(\nu, z) \\ F^{(2)}(\nu, \eta, z) \end{pmatrix} \equiv \mathrm{e}^{-i\mu_n^{(2)} z} \begin{pmatrix} \mathcal{A}_n^{(2)} \\ \mathcal{F}_n^{(2)}(\eta) \end{pmatrix}, \quad (38)$$

plugging it and integrating over θ we have

$$\left(-i\mu_n^{(2)} - i\Delta\nu k_u\right) A^{(2)}(\nu, z) = 2\pi\rho_1^3 \times$$
$$\times \int\int_0^z\int A^{(2)}(\nu_2, z_2)\, i\nu_2 (z_2 - z)\, \mathrm{e}^{-i\nu_2\phi_0(z_2 - z)} \times$$
$$\times \sum_m c_m^{(1)}(\nu_1, z_2)\, \mathcal{F}_m^{(1)}(\eta)\, \mathrm{e}^{-i\nu_1\phi_0(z_2 - z)} \times$$
$$\times\, \delta(\nu - \nu_1 - \nu_2)\, d\nu_2 dz_2 d\nu_1 d\phi, \quad (39)$$

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We assume that $A^{(2)}(\nu, z)$ is governed by a single frequency ν, this leads to $\nu_1 = 0$ and we have

$$\left(-i\mu_n^{(2)} - i\Delta\nu k_u\right) \mathrm{e}^{-i\mu_n^{(2)}z}\mathcal{A}_n^{(2)} = -2\pi\rho_1^3 \times$$
$$\times \int\int_0^z \mathrm{e}^{-i\mu_n^{(2)}z_2}\mathcal{A}_n^{(2)} i\nu (z_2 - z) \mathrm{e}^{-i\nu\phi(z_2-z)} \times$$
$$\times \sum_m c_m^{(1)}(0, z_2) \mathcal{F}_{m,\,\nu=0}^{(1)}(\eta)\, dz_2 d\phi, \qquad (40)$$

then we plug coefficients and eigenfunctions:

$$\left(\mu_n^{(2)} + \Delta\nu k_u\right) \mathrm{e}^{-i\mu_n^{(2)}z} = -2\pi\rho_1^3 \frac{eK[JJ]}{2\gamma_0^2 mc^2} 2k_u \times$$
$$\times \int\int_0^z \mathrm{e}^{-i\mu_n^{(2)}z_2} i\nu (z - z_2) \mathrm{e}^{i\nu 2k_u\eta(z-z_2)} \times$$
$$\times \sum_m \mathrm{e}^{-i\mu_{m,\,\nu=0}^{(1)}z_2} \left.\frac{\left[\Psi_m^\dagger, \Phi(0)\right]}{\left[\Psi_m^\dagger, \Psi_m\right]}\right|_{\nu=0} \left(\frac{\partial}{\partial\eta}F^{(0)}(\eta)\right) \times$$
$$\times \frac{\mathcal{A}_m^{(1)}}{\mu_m^{(1)}} dz_2 d\eta. \qquad (41)$$

As we see from (32) and (35) for $\nu = 0$ the dot products gives $A^{(0)}(0)$, which is often assumed to be zero. Plugging this, doing Laplace transform, and integrating over ϕ we get:

$$\frac{\mu_n^{(2)} + \Delta\nu k_u}{i\mu_n^{(2)} + s} = 2in_0\nu\pi\rho_1^3 \frac{eK[JJ]}{2\gamma_0^2 mc^2} 2k_u(-4k_u\nu) \times$$
$$\times \sum_m \frac{(2k_u\eta_0\nu + is)^{-3}}{i\left(\mu_n^{(2)} + \mu_{m,\,\nu=0}^{(1)}\right) + s} A^{(0)}(0) \frac{\mathcal{A}_m^{(1)}}{\mu_m^{(1)}}, \qquad (42)$$

and setting $s = 0$:

$$\frac{\mu_n^{(2)} + \Delta\nu k_u}{i\mu_n^{(2)}} = \frac{eK[JJ]}{\gamma_0^2 mc^2 \eta_0^3 k_u \nu} \sum_m \frac{-n_0\pi\rho_1^3 A^{(0)}(0)}{\mu_n^{(2)} + \mu_{m,\,\nu=0}^{(1)}} \frac{\mathcal{A}_m^{(1)}}{\mu_m^{(1)}}, \qquad (43)$$

We can further simplify the expression assuming zero detuning $\Delta\nu = 0$. For $A^{(0)}(0) = 0$ we have pure oscillatory solution $\mu_n^{(2)} = -\Delta\nu k_u$. However we cannot set $\eta_0 = 0$ to use simple solution for first order case, when we have $\mu_{m,\,\nu=0}^{(1)} = 0$, because we have η_0 in denominator. To use this simplification we can start from (41), integrating it over ϕ we have

$$\mu_n^{(2)} = -2\pi\rho_1^3 \frac{eK[JJ]}{2\gamma_0^2 mc^2} (2k_u\nu)^2 A^{(0)}(0) \times$$
$$\times \int_0^z \sum_m \mathrm{e}^{i\mu_n^{(2)}(z-z_2)} (z - z_2)^2 \mathrm{e}^{i\nu 2k_u\eta_0(z-z_2)} \frac{\mathcal{A}_m^{(1)}}{\mu_m^{(1)}} dz_2, \qquad (44)$$

where we set $\Delta\nu = 0$, then we change integration variable to $\tau = z - z_2$ and extend lower integration limit to $-\infty$, assuming that filed amplitudes grow exponentially [6]:

$$\mu_n^{(2)} = C \sum_m \frac{\mathcal{A}_m^{(1)}}{\mu_m^{(1)}} \int_{-\infty}^0 \mathrm{e}^{i\left(\mu_n^{(2)} + \nu 2k_u\eta_0\right)\tau} \tau^2 d\tau, \qquad (45)$$

where $\mathrm{Im}\left(\mu_n^{(2)} + \nu 2k_u\eta_0\right) < 0$ and C is a coefficient in front of the integral in (44), then we integrate

$$\mu_n^{(2)} = C \frac{2i}{\left(\mu_n^{(2)} + \nu 2k_u\eta_0\right)^3} \sum_m \frac{\mathcal{A}_m^{(1)}}{\mu_m^{(1)}}, \qquad (46)$$

then we set $\eta_0 = 0$:

$$\mu_n^{(2)} = \left(\frac{-i8\pi\rho_1^3 eK[JJ] k_u^2 \nu^2 A^{(0)}(0)}{\gamma_0^2 mc^2} \sum_m \frac{\mathcal{A}_m^{(1)}}{\mu_m^{(1)}}\right)^{\frac{1}{4}}. \qquad (47)$$

The relation between this solution and equation (42) and reasonable choice of s in (42) needs further analysis. Another possibility to deal with (41) is to set z equal to undulator length, which seems physically reasonable.

SUMMARY

In this article we studied the evolution of the initial density perturbation in an FEL, namely, we derived explicit formulas for the first order contribution and developed the method to compute the second order corrections. Following the similar procedure it is possible to compute the corrections of all orders. We plan to use this to compute the saturation effects in coherent electron cooler and estimate the maximum possible amplification. The method can be extended to the 3D case, as it was considered in [6], giving an integral equation for the field amplitude as a function of transverse radial coordinate. The difference from the method from [6] is that they made a certain assumption about higher order μ_n's, while we derived an equation for them.

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