THEORETICAL STUDY OF SMITH–PURCELL FREE-ELECTRON LASERS

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Abstract

We present an analytical theory for small-signal operation of a Smith–Purcell free-electron laser with a finitely thick electron beam travelling close to the surface of a grating. The dispersion equation is derived from a self-consistent set of small-signal equations describing the dynamics of beam-wave interaction. By solving the dispersion equation carefully, we reveal that the growth rate of the field amplitude holds a finite value at the Bragg point, which is different from previous theoretical predictions.

INTRODUCTION

It is believed that a compact, tunable, and coherent radiation source in the THz domain could be developed using the principles of Smith–Purcell free-electron lasers (SP-FEL)[1-5]. The principle of the beam-wave interaction above an open grating was established in Refs. [1-3], where the authors assumed a uniform electron beam filling the entire space above a lamellar grating and derived the dispersion relation for the evanescent wave on the surface of the grating. The authors pointed out that the device operates as a backward-wave oscillator (BWO) when the interaction with an electron beam occurs on the downward slope of the dispersion relation and as a travelling-wave tube (TWT) when the interaction occurs on the upward slope [2,3]. From their theory, they predicted that the spatial growth rate would be proportional to $I^{1/3} v_g^{-1/3}$, where I is the beam current and $v_g = d\omega/dk$ is the group velocity of the surface wave, and that the growth rate diverges at the Bragg point, where the group velocity vanishes. In evaluating the start current of the SP-FEL, all the authors followed the methods that have been used for analyzing BWOs. Almost the same boundary conditions were used in Refs. [2-4] to establish the equations determining the start current for SP-FEL. One condition, namely, the field vanishes at the downstream end of the grating, is used.

However, we know that it is possible for the surface wave to interact with the electron beam even at the Bragg point, so the prediction that the growth rate diverges at the Bragg point cannot be true. Also, the behavior that occurs at the ends of a grating, which will be addressed later, and the effect of this behavior in determining start current implies that the condition of a vanishing electromagnetic field at the downstream end is less reasonable. Therefore, it is necessary to reexamine the theoretical analysis.

DISPERSION

In the Cartesian coordinate system, the electrons initially move in the z direction with velocity v_0 in the vacuum above a lamellar grating along the trajectories $s \le x \le s + d$, and are coupled with the TM mode of an electromagnetic wave. The grating is ruled parallel to the y direction, and it has a period length L, groove width A, and groove depth H. The grating is assumed to be a perfect conductor, which means that the losses from the surface current can be ignored. The component of magnetic-flux density above the grating B_y can be expanded in the form

$$B_y = \sum_{p=-\infty}^{\infty} B_p(x) e^{-jk_p z}, \qquad (1)$$

Where, $k_p = k + 2\pi p/L$, and p is an integer. The wave equation is obtained below:

$$\frac{\partial^2}{\partial x^2} B_p(x) - \left(k_p^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2}\right) B_p(x) = 0 \qquad (2)$$

Following the methods using in Ref.[2,3], it is straightforward to get the dispersion equation

$$\sum_{p=-\infty}^{\infty} \Re_p \frac{\alpha_{g,0} \tan(\alpha_{g,0} d) S_{1,p} S_{2,p}}{LA\alpha_p} = 1, \qquad (3)$$

Where,

$$\Re_p = \frac{1+g_p}{1-g_p}, \qquad S_{1,q} = \int_0^A e^{jk_q z} dz$$

$$g_p = \frac{(e^{2\alpha_{II} d} - 1)(\alpha_p^2 \varepsilon_z^2 (k_p v_0 \varepsilon_x - \omega)^2 - \alpha_{II}^2 (v_0 k_p - \omega)^2)}{e^{2\alpha_p s}((\alpha_p \varepsilon_z (k_p v_0 \varepsilon_x - \omega) + \alpha_{II}(v_0 k_p - \omega))^2 e^{2\alpha_{II} d} - (\alpha_p \varepsilon_z (k_p v_0 \varepsilon_x - \omega) - \alpha_{II}(v_0 k_p - \omega))^2)}$$

$$S_{2,p} = \int_0^A e^{jk_p z} dz, \qquad \varepsilon_x = \frac{\omega - v_0 k_p - \dfrac{v_0 \omega_p^2}{c^2 k_p}}{\omega - v_0 k_p - \dfrac{\omega_p^2}{\omega}}$$

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$$\varepsilon_z = 1 - \frac{\omega_p^2}{\omega(\omega - v_0 k_p)}, \quad \alpha_p^2 = k_p^2 - \omega^2/c^2$$

$$\alpha_{II}^2 = k_p^2 - \omega^2/c^2 + \omega_p^2/c^2, \quad \alpha_{g,0} = \omega/c$$

In the absence of an electron beam, ω_p vanishes and the dispersion equation is simplified as

$$\sum_{p=-\infty}^{\infty} \frac{\alpha_{g,0} \tan(\alpha_{g,0} d) S_{1,p} S_{2,p}}{LA\alpha_p} = 1. \quad (4)$$

The grating parameters used in this paper are set to be L=173μm, A=62μm, H=100μm, and the period number of 73. Using these parameters, the dispersion relation is obtained by numerically solving Eq. (4), and the result is shown in Fig. 1, which shows that the operating point (ω_0, k_0) of the laser is where beam line βk intersects the dispersion curve. It also shows that for electrons with energy of 90 keV, the intersection occurs on the downward slope. We carried $-4 \le p \le 4$ in the expansion of Eq. (4), which showed good convergence.

We know that a surface mode consists of the superposition of an infinite number ($p = -\infty .. \infty$) of spatial harmonics, and that those with positive k_p carry

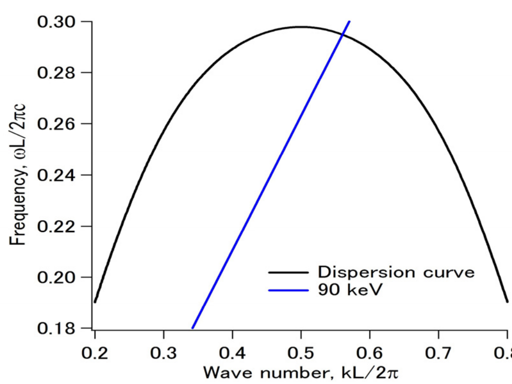

Figure 1: Dispersion relation of the surface wave.

energy flow forward, while those with negative k_p carry energy flow backward. From the dispersion equation, we can calculate the average power carried by each harmonic with the average power

$$S_p = \int_0^\infty dx \frac{1}{2} E_{x,p} H_{y,p}^*,$$

SPATIAL GROWTH RATE

Usually, Eq. (3) can be simplified as discussed below. We know that only the zero[th] space harmonic can synchronize with the electron beam; thus, it is reasonable to only consider $\Re_{p=0}$ near the operating point (ω_0, k_0). Also, if the beam density is small enough, $\Re_{p=0}$ can be expanded in powers of small magnitude $\frac{\omega_b^2}{\gamma^3(\omega - kv_0)^2}$, where $\omega_b^2 = \gamma\omega_p^2$. Then, Eq. (3) can be simplified to be

$$\xi - 1 + \Theta\Omega = 0 \quad (5)$$

where

$$\xi = \sum_{p=-\infty}^{\infty} \frac{\alpha_{g,0} \tan(\alpha_{g,0} d) S_{1,p} S_{2,p}}{LA\alpha_p}, \quad \Omega = \frac{\alpha_{g,0} \tan(\alpha_{g,0} d) S_{1,0} S_{2,0}}{LA\alpha_0},$$

$$\Theta = \sum_{n=1}^{\infty} \frac{X^n \Re_{p=0}^{(n)}(0)}{n!}, \quad X = \frac{\omega_b^2}{\gamma^3 v_0^2 \delta k^2}, \quad \delta k = k - k_0,$$

and $\Re_{p=0}^{(n)}(0)$ is the n[th] derivative of $\Re_{p=0}$ at 0. Note that the beam density should satisfy the conditions $\omega_b / \omega\sqrt{\gamma} << 1$ and $\omega_b^2/(\gamma^3 v_0^2 \delta k^2) << 1$ in the simplifying process.

Next, we can expand ξ at the operating point (ω_0, k_0). If we take the first order of expanded powers,

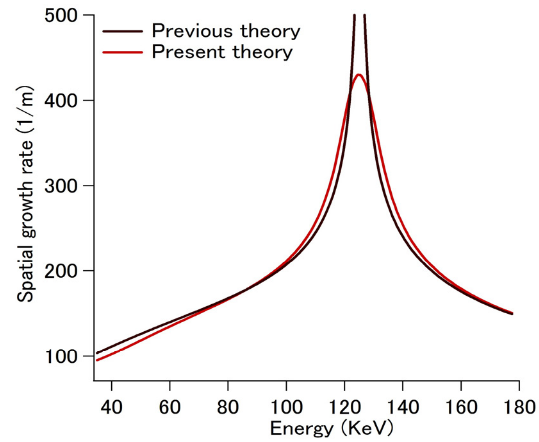

Figure 2: Spatial growth rate.

Eq. (3) can be written as

$$\xi'(\omega_0, k_0)\delta k + (e^{-2\alpha_0 d} - 1)e^{-2\alpha_0 s}\Omega \frac{\omega_b^2}{\gamma^3 \beta^2 c^2 \delta k^2} = 0,$$

where $\xi'(\omega_0, k_0)$ is the derivative $\partial\xi/\partial k$ at (ω_0, k_0). From the above equations, obtaining the spatial growth rate is straightforward:

$$\mu = \mathrm{Im}(\delta k) = \frac{\sqrt{3}}{2}\left| -\frac{(e^{-2\alpha_0 d} - 1)e^{-2\alpha_0 s}\Omega\omega_b^2}{\gamma^3 \beta^2 c^2 \xi'(\omega_0, k_0)} \right|^{\frac{1}{3}}, \quad (6)$$

Eq. (6) shows that the growth rate diverges at the Bragg point where $\xi'(\omega_0, k_0)$ vanishes. However, this is not due to the actual physics that occurred there but is instead due to the rough mathematical calculation. If we take into account more terms of expansion, the growth rate would have a finite value at any point on the dispersion curve. We try to attain better precision with fewer terms of expanded powers. In this paper, we expand ξ and Θ up to the second order; thus, Eq. (5) should be rewritten as

$$\sum_{n=1}^{2} \frac{\delta k^n \xi^{(n)}(\omega_0, k_0)}{n!} + \Omega \cdot \sum_{n=1}^{2} \frac{X^n \Re_{p=0}^{(n)}(0)}{n!} = 0, \quad (7)$$

This equation can be numerically solved to obtain δk, and $\mathrm{Im}(\delta k)$ would be the spatial growth rate . Eq.(7) has six solutions. One of them that has the maximum

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imaginary part indicates the mainly growing mode, and the other solutions can be neglected.

We assume that the electron beam fills a region of width d=24μm. Beam height above the grating and the beam energy are assumed to be s=30μm and E_b=90 keV, respectively. In these calculations, the beam current is fixed to $I=1$ mA, and we have the relation $\omega_b^2 = Ie_0 / m_0\varepsilon_0\beta c d^2$, where m_0 is the electron mass, d is the thickness and width of the electron beam. The results are shown in Fig. 2, where the spatial growth rate is a function of beam energy. As is shown in Fig. 2, according to the previous theory, the growth rate diverges when the electron beam energy reaches 124 keV, i.e., the Bragg point; in contrast, our theory gives a finite value.

START CURRENT AND TEMPORAL GROWTH RATE

In our theory, the mechanism of beam-wave interaction should occur like this: the zeroth order space harmonic interacts with the electron beam moving in the z direction and the whole harmonics are amplified during the interaction; at the downstream end, the $p \geq 0$ harmonics go out the grating (where they are partially reflected and partially diffracted; however, we ignore these reflections in the present theory), while the $p < 0$ harmonics are retained. Note that they are retained but not *reflected,* because $p < 0$ harmonics intrinsically move in the –z direction; energy carried by $p < 0$ harmonics are reapportioned among the whole harmonics to satisfy the boundary condition on the surface of a grating; in addition, at the upstream end, $p \geq 0$ harmonics are retained, and they start the second round trip. Because the zeroth and –1st order harmonics hold most of the energy of the surface wave and because they are faster than the other space harmonics, it is reasonable to consider only these two harmonics in the following analysis.

We define the total power flow of the whole space harmonics as $S_{total} = \sum_{p=-\infty}^{\infty} |S_p|$. The ratio of a given harmonic to the total power flow is written as $\rho_p = |S_p| / S_{total}$. Thus, the condition for device to start oscillating should be $e^{2\mu\ell} \cdot \rho_{-1} \cdot \rho_0 = 1$, where ℓ is the total length of a grating. From the no-beam dispersion equation, we can calculate ρ_p at operating point (ω_0, k_0); thus the required spatial growth rate for starting oscillation can be acquired. Then, we can work out the start current with the help of Eq. (7).

In a round trip, the fields are amplified only when the zeroth order harmonic moves forward with the electron beam; therefore, the gain of the field can be written as a function of time $e^{(\mu\ell + \frac{1}{2}\ln(\rho_0\rho_{-1}))\frac{t}{t_e}}$, where t_e is the effective time. In addition, we know that the energy is brought back to the upstream end mainly by the –1st order harmonic, so the relation between effective time t_e and real time t can be easily worked out as $t_e = \frac{|v_{-1}| + v_0}{|v_{-1}| \cdot v_0}\ell$.

Here, v_p is phase velocity for a p^{th} order harmonic and also the energy velocity of the harmonic. Finally, we get the temporal growth rate as

$$\sigma = (\mu + \frac{1}{2\ell}\ln(\rho_{-1}\rho_0)) \frac{v_0 |v_{-1}|}{|v_{-1}| + v_0}. \quad (8)$$

Using the parameters mentioned above, we calculated the temporal growth rates as function of beam current, and the result is given in Fig. 3, where the start current is also shown.

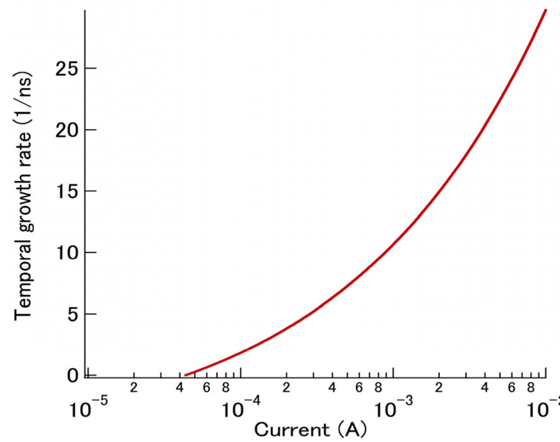

Figure 3: Temporal growth rate and start current.

CONCLUSION

In conclusion, by carefully processing the dispersion equation, we find that the growth rate does not diverge near the Bragg point, which is more reasonable than the previous theory. We develop a simple method to evaluate the start current based on the power flow of space harmonics, and also provide a way to convert spatial growth rate to temporal growth rate.

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