THEORY OF THE QUANTUM FEL IN A NUTSHELL

P. Preiss*, Helmholtz-Zentrum Dresden-Rossendorf eV, D-01328 Dresden, Germany
and Institut für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany
R. Sauerbrey, Helmholtz-Zentrum Dresden-Rossendorf eV, D-01328 Dresden, Germany
M. S. Zubairy, Institute for Quantum Science and Engineering,
Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA
R. Endrich, E. Giese, P. Kling, M. Knobl and W. P. Schleich,
Institut für Quantenphysik and Center for Integrated Quantum Science and Technology (IQ^{ST}),
Universität Ulm, D-89069 Ulm, Germany

Abstract

A new regime of the free-electron laser arises when the recoil of the electron due to its scattering in the wiggler and laser field cannot be neglected any more. In such a quantum free-electron laser the discreetness of the momenta becomes visible and leads to novel effects. We present a quantum mechanical theory in this domain.

INTRODUCTION

Classical electrodynamics combined with classical statistical mechanics are sufficient [1] to describe today's free-electron laser (FEL) devices. However, new developments in accelerator and laser physics raise hope for the experimental realization of a quantum free-electron laser (QFEL). In this paper we outline a theory of the QFEL guided by techniques of quantum optics developed in the context of the one-atom maser [2]. For an earlier and different approach towards the QFEL we refer to [3][4].

We start by recalling the Hamiltonian [5] corresponding to a one-dimensional, single-particle description of the FEL in the co-moving Bambini-Renieri frame where the electrons move with a non-relativistic velocity. Based on the classical free-electron laser (CFEL) and introducing recoil effects by hand we illustrate the emergence of the quantum mechanical regime of the FEL from the pendulum equation. Moreover, a perturbation theory in which these effects are fully taken into account illustrates in a vivid way the characteristic features of the QFEL. Here, the electron can emit or absorb only a single photon leading to a two-level dynamics. In this way we obtain an analytically solvable model for the QFEL. The resulting gain function confirms the applicability of the two-level QFEL model. Finally, we establish the key experimental requirements for realizing a QFEL device. Our conditions are in agreement with the ones put forward in [4].

QUANTUM MODEL OF FEL

The interaction of a single electron in the FEL with the wiggler field and the emitted laser field, when described in the co-moving Bambini-Renieri frame where the frequencies $\omega = ck$ of both fields coincide, is determined [5] by the Hamiltonian $\hat{H} \equiv \hat{H}_\mathrm{F} + \hat{H}_\mathrm{I}$ consisting of the free part

$$\hat{H}_\mathrm{F} \equiv \frac{\hat{p}^2}{2m} + \hbar\omega \left(\hat{a}_\mathrm{L}^\dagger \hat{a}_\mathrm{L} + \hat{a}_\mathrm{W}^\dagger \hat{a}_\mathrm{W}\right) \tag{1}$$

and the interaction part

$$\hat{H}_\mathrm{I} \equiv \hbar\tilde{g}\left(\hat{a}_\mathrm{L}^\dagger \hat{a}_\mathrm{W} e^{-i2k\hat{z}} + \hat{a}_\mathrm{L}\hat{a}_\mathrm{W}^\dagger e^{i2k\hat{z}}\right) \tag{2}$$

with the coupling strength $\tilde{g}$. The creation or annihilation operators $\hat{a}_\mathrm{L}^\dagger$ or $\hat{a}_\mathrm{L}$ and $\hat{a}_\mathrm{W}^\dagger$ or $\hat{a}_\mathrm{W}$ of the laser and wiggler field, respectively, obey the familiar commutation relation $[\hat{a}_j, \hat{a}_j^\dagger] = 1$, where $j = \mathrm{L}$ and W. Moreover, $\hat{z}$ and $\hat{p}$ denote the position and momentum operator of the electron in the Bambini-Renieri frame obeying $[\hat{z}, \hat{p}] = i\hbar$ and $m \equiv m_e(1 + a_0^2)$ represents the shifted mass of the electron where a_0 is the wiggler parameter, $k \equiv 2\pi/\lambda$ the wave number of the laser and wiggler field with wavelength λ and m_e the rest mass of the electron. The exponential $e^{\pm i2k\hat{z}}$ in Eq. (2) indicates a shift in the momentum of the electron by the recoil $q \equiv 2\hbar k$ when one photon of the wiggler scatters into the laser field or vice versa.

In the interaction picture $\hat{H}$ transforms into

$$\hat{H}_\mathrm{I}^{(\mathrm{I})} \equiv \hbar g \left(\hat{a}_\mathrm{L}^\dagger e^{-i2k\hat{z}} e^{-i2k(\hat{p}-q/2)t/m} + \mathrm{h.c.}\right) \tag{3}$$

where we have used the semiclassical approximation $\hat{a}_\mathrm{W}^\dagger \approx \hat{a}_\mathrm{W} \approx \sqrt{n_\mathrm{W}}$ for the wiggler field.

Assuming circularly polarized vector potentials

$$\widehat{\mathbf{A}}_\mathrm{L} \equiv \mathcal{A}_\mathrm{L}\left(\mathbf{e}\hat{a}_\mathrm{L} e^{-i(\omega t - k\hat{z})} + \mathrm{h.c.}\right) \tag{4}$$

and

$$\widehat{\mathbf{A}}_\mathrm{W} = \mathcal{A}_\mathrm{W}\left(\mathbf{e}\hat{a}_\mathrm{W} e^{-i(\omega t + k\hat{z})} + \mathrm{h.c.}\right) \tag{5}$$

for the laser field, and the wiggler field in Weizsäcker-Williams approximation of strengths $\mathcal{A}_\mathrm{L}$ and $\mathcal{A}_\mathrm{W}$, respectively, and polarization vectors that obey the relation $\mathbf{e}^2 = \mathbf{e}^{*2} = 0$, we obtain the coupling constant

$$g = \frac{e_0^2}{\hbar m}\mathcal{A}_\mathrm{L}\mathcal{A}_\mathrm{W}\sqrt{n_\mathrm{W}} \tag{6}$$

with the elementary charge e_0.

* p.preiss@hzdr.de

ISBN 978-3-95450-123-6

CFEL VERSUS QFEL

The Hamiltonian of the CFEL in the low gain regime which follows from $\hat{H}$ in the Schrödinger picture when we turn all operators into c-numbers and neglect the number n_{L} of photons of the laser in Eq. (1) reads

$$H_{\mathrm{cl}} \equiv \frac{p^2}{2m} + 2\hbar g\sqrt{n_{\mathrm{L}}}\cos(2kz). \tag{7}$$

Using the Hamilton equations $\dot{z} = \partial H_{\mathrm{cl}}/\partial p$ and $\dot{p} = -\partial H_{\mathrm{cl}}/\partial z$ the dynamics of the ponderomotive phase $\phi \equiv 2kz + \pi$ is determined by a pendulum equation

$$\ddot{\phi} + \Omega^2 \sin\phi = 0 \tag{8}$$

with frequency $\Omega = \sqrt{8k^2\hbar g\sqrt{n_{\mathrm{L}}}/m}$, in agreement with [1] where Eq. (8) is derived in the laboratory frame.

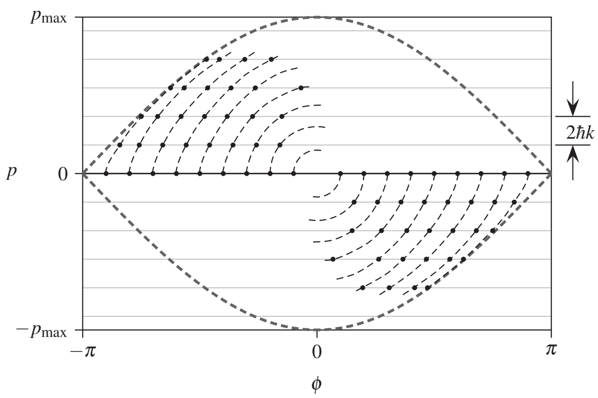

Figure 1: Transition from the classical to the quantum FEL illustrated in phase space: In the classical regime the trajectories of the FEL are continuous. In the quantum regime the discreetness of the momentum states manifests itself and the trajectories become discontinuous.

By definition the classical theory lacks the possibility of including the recoil of the electron and we have to introduce it by hand as illustrated in Fig. 1. Indeed, due to energy and momentum conservation in the emission or absorption process of a laser photon the momentum of the electron changes by $\pm 2\hbar k$. Hence, the motion of the electron in phase space cannot be continuous as predicted by the pendulum equation (8). Discrete momentum states occur as indicated in Fig. 1 when the recoil is not negligible any more. The regime of the QFEL is reached when only two distinct momentum states are allowed in phase space. This condition reduces to $q > p_{\max}$ where the maximally achievable momentum $p_{\max}$ is determined by

$$\frac{p_{\max}^2}{2m} = H_{\mathrm{sep}} = 2\hbar g\sqrt{n_{\mathrm{L}}}. \tag{9}$$

Here H_{sep} is the energy of an electron moving along the separatrix of the classical Hamiltonian H_{cl}.

The QFEL condition $q > p_{\max}$ finally translates into the condition

$$\alpha \equiv \frac{2\hbar g\sqrt{N}}{q^2/2m} < 1 \tag{10}$$

for the QFEL parameter α with the maximal photon number N.

REACHING THE TWO-LEVEL LIMIT

The distinction between the CFEL and QFEL stands out most clearly in the diagonal elements $W_n(t)$ of the density matrix of the laser field describing its photon statistics. Indeed, their time dependence [6]

$$W_n(t+\tau) = \sum_l W_l(t)\left|\langle n, p_0 + (l-n)q\,|\hat{\mathcal{U}}(\tau)|\, l, p_0\rangle\right|^2 \tag{11}$$

follows from the time evolution operator

$$\hat{\mathcal{U}}(\tau) \equiv T\left[\exp\left(-\frac{i}{\hbar}\int_0^\tau dt\,\hat{H}_{\mathrm{I}}^{(\mathrm{I})}\right)\right] \tag{12}$$

with the Dyson time-ordering operator T and the interaction time τ of the electron with the field. The states $|\,n, p\rangle$ are the eigenstates of the photon number operator $\hat{a}_{\mathrm{L}}^\dagger \hat{a}_{\mathrm{L}}$ of the laser field and the momentum operator $\hat{p}$, and p_0 denotes the initial momentum.

Evaluating $\hat{\mathcal{U}}$ up to second order in g and using the coarse-grain derivative the master equation reads

$$\dot{W}_n = r(g\tau)^2\Big[(n+1)\mathcal{S}_+^2 W_{n+1} + n\mathcal{S}_-^2 W_{n-1} - \left((n+1)\mathcal{S}_-^2 + n\mathcal{S}_+^2\right)W_n\Big], \tag{13}$$

where we have introduced the electron injection rate $r \equiv N_{\mathrm{el}}/\tau$ [6] and the functions

$$\mathcal{S}_\pm \equiv \frac{\sin\left(k(p \pm q/2)\tau/m\right)}{k(p\pm q/2)\tau/m} \tag{14}$$

crucially determine the dynamics of the photon statistics. Indeed, for

$$\frac{kq\tau}{m} > 1 \tag{15}$$

the functions $\mathcal{S}_\pm$ are well separated as indicated in Fig. 2. In this case each momentum of the electron is connected to the emission or absorption of a single laser photon. Starting with an initial momentum $p_0 \approx q/2$ only an emission driven by a non-vanishing value of $\mathcal{S}_-$ will occur. After the emission of one photon the electron recoils by an amount q and the probability of emitting another photon becomes approximately zero.

We note that the condition (15) was already obtained in [7] by performing perturbation theory in the classical regime. However, in our approach the recoil is fully taken into account.

ISBN 978-3-95450-123-6

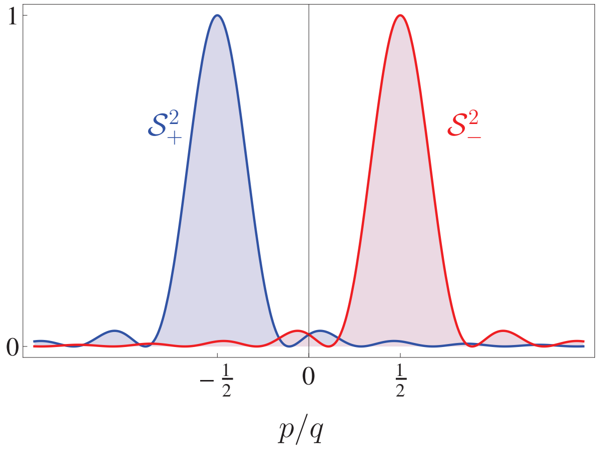

Figure 2: Separation of momentum domains corresponding to absorption and emission of laser photons defining the QFEL illustrated here for $kq\tau/m = 8$. Due to the recoil giving rise to well-separated functions $\mathcal{S}_\pm$ the electron cannot absorb or emit more than one laser photon.

RABI OSCILLATIONS IN THE QFEL

An analytically solvable model for the QFEL illustrated in Fig. 3 and based on only two momentum states emerges from the superposition

$$|\Psi(t)\rangle \equiv \psi_e(t)\,|e\rangle + \psi_g(t)\,|g\rangle \tag{16}$$

of the dressed excited state $|e\rangle \equiv |n_0, p_0\rangle$ and the dressed ground state $|g\rangle \equiv |n_0+1, p_0-q\rangle$ with the initial number of photons n_0 in the laser field.

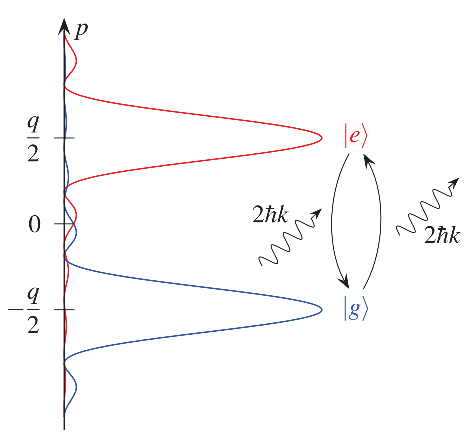

Figure 3: Two-level model of the QFEL: Whereas in an ordinary laser the states of the medium are determined by the internal degrees of freedom, in the FEL this role is played by the momentum states. In the limit of the QFEL the well-separated functions $\mathcal{S}_\pm$ ensure that mainly the two momentum states $|q/2\rangle$ and $|-q/2\rangle$ of the electron take part in the lasing process corresponding the creation of a laser photon at the expense of a wiggler photon giving rise to a total momentum change of $2\hbar k$.

We choose the initial momentum $p_0 \approx q/2$ of the electron as to minimize the phase in the second exponent in Eq. (3). Thus, we start in the regime where $\mathcal{S}_-$ determines the interaction in the FEL and other momenta correspond to rapidly oscillating phases provided condition (15) is satisfied.

In the spirit of the rotating wave approximation we assume that the phases $\sim kq/m$ are larger than the coupling strength $g\sqrt{n_0}$ which leads us again to the condition $\alpha < 1$. This argument is another justification of the two-level ansatz (16) for the QFEL.

Inserting Eq. (16) into the Schrödinger equation

$$i\hbar\frac{d}{dt}\,|\Psi\rangle = \hat{H}_\mathrm{I}^{(\mathrm{I})}\,|\Psi\rangle \tag{17}$$

yields coupled equations of motion for the amplitudes ψ_e and ψ_g with the solutions

$$\psi_e(t) = e^{i\delta t/2}\left[\cos(\lambda t) - i\frac{\delta/2}{\lambda}\sin(\lambda t)\right] \tag{18}$$

and

$$\psi_g(t) = -ie^{-i\delta t/2}\frac{g\sqrt{n_0+1}}{\lambda}\sin(\lambda t), \tag{19}$$

where we have introduced the detuning parameter $\delta \equiv 2k(p_0 - q/2)/m$.

Therefore, the dynamics of the QFEL consist of Rabi oscillations between the dressed states $|g\rangle$ and $|e\rangle$ with the QFEL frequency

$$\lambda \equiv \sqrt{g^2(n_0+1) + (\delta/2)^2}. \tag{20}$$

Starting with $p_0 \approx -q/2$ also leads us to a two-level system with Rabi oscillations between $|g\rangle \equiv |n_0, p_0\rangle$ and $|e\rangle \equiv |n_0 - 1, p_0 + q\rangle$ with amplitudes $\tilde{\psi}_e$ and $\tilde{\psi}_g$. The detuning is now given by $\tilde{\delta} \equiv 2k(p_0 + q/2)/m$ and the Rabi frequency reads $\tilde{\lambda} \equiv \sqrt{g^2 n_0 + \left(\tilde{\delta}/2\right)}$.

QFEL GAIN CURVE

The classical gain curve of the FEL in the small signal regime is given by Madey's theorem [8]. In our model we express the gain function

$$G \equiv \sum_{n=-\infty}^{\infty} nP_n(n_0, p_0) \tag{21}$$

in terms of the probabilities P_n for scattering n photons from the wiggler into the laser field, or vice versa in case of negative values of n.

The QFEL model given by (16) yields the gain function

$$G_\mathrm{QFEL}(\tau) = \begin{cases} |\psi_g(\tau)|^2 & \text{for } p_0 \in (0, q) \\ -\left|\tilde{\psi}_e(\tau)\right|^2 & \text{for } p_0 \in (-q, 0) \\ 0 & \text{else} \end{cases} \tag{22}$$

for the interaction time τ.

It is interesting to compare this expression and

ISBN 978-3-95450-123-6

$$G_{\rm PT} \equiv n_0(g\tau)^2\left(\mathcal{S}_-^2 - \mathcal{S}_+^2\right) + (g\tau)^2\mathcal{S}_-^2 \tag{23}$$

obtained with the help of Eq. (13) in perturbation theory up to second order in g, to the gain function determined by a simulation of the evolution of the transition probabilities P_n. Here, we take into account the dressed states $|n_0 + l, p_0 - lq\rangle$ with $|l| \leq 5$. In Fig. 4 we compare the so-calculated gain functions for $\alpha = 0.2$ and the parameters $n_0 = 10$, $k = \gamma 2\pi/\lambda_{\rm W}$ with $\gamma = 100$ and wiggler wavelength $\lambda_{\rm W} = 10^{-6}$ m. Here, we set the maximal photon number $N = n_0 + 5$ and the interaction time $\tau = \pi/2g\sqrt{n_0+1}$ corresponding to half of the Rabi oscillation at resonance $p_0 = q/2$. This figure clearly demonstrates the validity of our two-level model for the QFEL.

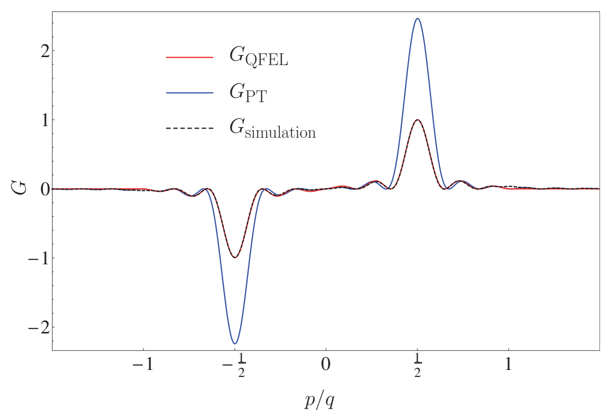

Figure 4: Confirmation of the two-level model of the QFEL by numerical simulation of the gain function for a small QFEL parameter $\alpha = 0.2$.

EXPERIMENTAL REQUIREMENTS

A comparison of the parameters of our model with the corresponding quantities of the classical theory [1] connects the QFEL parameter

$$\alpha = 4\left(\frac{\gamma m c}{\hbar k_{\rm L}}\rho_{\rm FEL}\right)^{3/2} \tag{24}$$

with the Pierce parameter [10]

$$\rho_{\rm FEL} = \frac{1}{2\gamma}\left(\frac{J_e\lambda_{\rm W}^2 a_0^2}{I_A 2\pi}\right)^{2/3}. \tag{25}$$

Here, J_e and $I_A = 17.045$ kA denote the peak current density of the electron beam and the Alfvén current, respectively, together with the dimensionless energy $\gamma \equiv E/m_e c^2$ of the electron and the wave number $k_{\rm L} = 2\pi/\lambda_{\rm L}$ of the emitted radiation. We note that in terms of these quantities the QFEL condition (10) translates into the one proposed in [3].

Assuming a laser wiggler with wavelength $\lambda_{\rm W}$ and intensity I_0 we obtain the dimensionless wiggler parameter $a_0 = 0.85 \times 10^{-9}\lambda_W[\mu{\rm m}]I_0^{1/2}[{\rm W/cm^2}]$ [9] and furthermore $k_{\rm L} = 4\gamma^2 k_{\rm W}/(1+a_0)^2$. Hence, the QFEL condition (10) takes the form

$$J_e < \frac{\gamma^6}{\lambda_{\rm W}^7}\frac{6.938\times 10^{-19}\,{\rm AWm^3}}{I_0\left[1 + 7.2\times 10^{-11}\,{\rm W^{-1}}\lambda_{\rm W}^2 I_0\right]^3} \tag{26}$$

connecting the key experimental quantities J_e, γ, $\lambda_{\rm W}$ and I_0.

Further requirements, for instance on the emittance of the electron beam pointed out in [4], are in full accordance with this model.

SUMMARY

In conclusion, we have developed a theory of the QFEL. Starting from the classical pendulum equation we have derived the QFEL condition and have shown the characteristic two-level dynamics. We have obtained the gain functions in three different ways and have demonstrated the applicability of our QFEL model. Moreover, we have established the crucial experimental requirements for realizing a QFEL device.

ACKNOWLEDGMENT

We thank W. Becker, M. Bussmann, T. E. Cowan, G. Kurizki, M. O. Scully and S. Varro for many fruitful discussions on the classical and the quantum aspects of the FEL.

REFERENCES

[1] P. Schmüser et al., *Ultraviolet and Soft X-Ray Free-Electron Lasers*, (Berlin: Springer, 2009).

[2] W. P. Schleich, *Quantum Optics in Phase Space*, (Berlin: Wiley-VCH, 2001).

[3] R. Bonifacio, N. Piovella and G. R. M. Robb, Fortschr. Phys. 57 (2009) 1041.

[4] R. Bonifacio, N. Piovella, M. M. Cola and L. Volpe, Nucl. Instr. and Meth. A 577 (2007) 745.

[5] W. Becker and M. S. Zubairy, Phys. Rev. A 25 (1982) 2200.

[6] W. Becker, M. Scully and M. S. Zubairy, in *Coherence and Quantum Optics V*, L. Mandel and E. Wolf, eds., (1984) 818.

[7] W. Becker and J. K. McIver, Physics Reports 154 (1987) 205.

[8] W. Becker and J. K. McIver, Z. Phys. D 7 (1988) 353.

[9] A. D. Debus, M. Bussmann, M. Siebold, A. Jochmann, U. Schramm, T. E. Cowan and R. Sauerbrey, Appl. Phys. B 100 (2010), 61.

[10] M. Xie, in *Proceedings of the Particle Accelerator Conference*, Dallas (1995), 183.

ISBN 978-3-95450-123-6