COLLECTIVE AND INDIVIDUAL ASPECTS OF FLUCTUATIONS IN RELATIVISTIC ELECTRON BEAMS FOR FREE-ELECTRON LASERS*

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Abstract

Fluctuations in relativistic electron beams for free-electron lasers (FELs) exhibit both collective and individual particle aspects, similar to that seen in non-relativistic plasmas. We show that the density fluctuations are described by a linear combination of the collective plasma oscillation and the random individual motion of Debye-screened dressed particles. The relative importance of the individual to the collective motion is determined by comparing the fluctuation length scale divided by two pi with the relativistic beam Debye length. Taking into account the fact that the velocity spread is caused by both the energy spread and the angular divergence, we derive a simple formula for the minimum value of the Debye length using a solvable 1-D model. For electron beams used for x-ray self-amplified spontaneous emission (SASE) we find that the Debye length is comparable to the radiation wavelength, and that therefore the collective motion is not relevant..

INTRODUCTION

Shot noise in relativistic electron beams has its origins in the discrete nature of the electron: over sufficiently small time and length scales the current will fluctuate due to variations in the number of electrons measured. However, the term "shot noise" has a more precise definition with more restrictive properties than being merely the fluctuations associated with discrete particles. Specifically, shot noise is connected with a Poisson process; in an electron beam this means that while the average flow is given by the current, the arrival time of any particular electron is independent of the arrival times of all other electrons, and the characteristic fluctuations in number N scales as $\delta N/N \sim 1/\sqrt{N}$. To be more explicit, we define the time of the j^{th} electron in the bunch as $\zeta_j(z) \equiv z - c\beta_0 t_j$, where z and t_j are the longitudinal coordinate and particle time, respectively, while β_0 is the reference velocity scaled by the speed of light c. If we consider the spectral content of the density fluctuations or bunching given by

$$b_k(z) \equiv \frac{1}{N}\sum_{j=1}^{N} e^{-ik\zeta_j(z)}, \tag{1}$$

than the ensemble average $\langle b_k(z)\rangle = 0$, while the average value of $\langle |b_k(z)|^2\rangle = 1/N$ for a beam that is characterized by shot noise.

The equilibrium fluctuations of an ideal gas are those of shot noise. Additionally, the emission processes of excited atoms, diodes, and cathodes is typically assumed to be Poisson, so that laser light, electric currents, and particle beams are produced with shot-noise type fluctuations. However, inter-particle forces can induce correlations that can change the fluctuation statistics from that of shot noise. For example, the ponderomotive force in an undulator tends to increase any density fluctuations near the resonant wavelength, which is the origin of the free-electron laser (FEL) instability and self-amplified spontaneous emission (SASE). On the other hand, electrostatic repulsion tends to smooth out density fluctuations. This physical effect has been given the name "shot noise suppression" in the FEL literature [1, 2, 3].

Due to the long-range electrostatic force, the equlibrium density fluctuations in a plasma are quite different than those of shot noise. We can understand this by considering a plasma of density n_0, whose natural plasma frequency is $\omega_p \equiv \sqrt{e^2 n_0/\epsilon_0 m}$, where e is the magnitude of the electron charge, m is its mass, and ϵ_0 is the vacuum dielectric constant. If we assume that the rms thermal spread is σ_v, we can define the Debye length $\lambda_D \equiv \sigma_v/\omega_p$ which is the distance a particle moves due to its thermal motion during one plasma oscillation. For distances less than $2\pi\lambda_D$, the particle is not aware of the plasma, and individual particle interactions dominate the dynamics. Conversely, the plasma can organize itself so as to screen any individual test particle over length-scales much longer than $2\pi\lambda_D$, so that the long-scale physics is characterized by collective motion of the particles [the plasma (or Langmuir) oscillation]. To be more quantitative, Rostoker has shown in an elegant calculation that in a Maxwellian plasma the ensemble averaged bunching squared is given by [4]

$$\left\langle |b_k|^2 \right\rangle = \frac{1}{N}\frac{(k\lambda_D)^2}{1+(k\lambda_D)^2}. \tag{2}$$

In the limit $k\lambda_D \ll 1$, $\langle |b_k|^2\rangle \to (k\lambda_D)^2/N$, and we see that the correlations have reduced the density fluctuations far below that of shot noise. The electrostatic energy associated with these density fluctuations can be shown to be $k_B\sigma_v^2/2$ per mode as one would expect from equipartition (k_B is Boltzmann's constant). In the other limit, when $k\lambda_D \gtrsim 1$ the fluctuations in the bunching approach that of shot noise, $\langle |b_k|^2\rangle \approx 1$.

In the following we review a simple non-equilibrium model for the electron beam developed in [5, 6] which shares certain characteristics with classical plasmas. We compute the fluctuation characteristics and compare these to simple simulation results. We then compute the relevant Debye length, and show that collective oscillations are not relevant for electron beams suitable for x-ray generation.

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FLUCTUATIONS IN RELATIVISTIC ELECTRON BEAMS

While an electron beam can be considered a non-neutral plasma, it is often not in equilibrium. In fact, electron beams are created having shot noise statistics due to the fact that emission from a cathode is a Poisson process, and typically one rapidly accelerates the beam to high energies so as to "freeze out" the electrostatic repulsion. For this reason, we will consider a simple initial value problem for the electron beam in the presence of the electrostatic force that was first introduced in [5, 6]. Recently, the 3D initial value problem including emittance effects and betatron oscillations was presented in [7]. Our goal here is rather modest, however: we merely want to point out the limitations of the collective response at x-ray wavelengths, so that we have decided to sacrifice the added generality of [7] in the interest of greater mathematical simplicity.

To begin, we must first find the canonical conjugate to the particle time ζ_j, which we define to be the velocity difference $v_j \equiv d\zeta_j/dz = 1 - \beta_0/\beta$. We furthermore introduce the particle distribution function $f(\zeta, v; z)$ over the phase space (ζ, v), which satisfies

$$\frac{\partial f}{\partial z} + v\frac{\partial f}{\partial \zeta} - \frac{eE}{mc^2\beta_0\gamma_0^3}\frac{\partial f}{\partial v} = 0. \tag{3}$$

Here $\gamma_0 \equiv (1-\beta_0^2)^{-1/2}$ is the Lorentz factor of the reference particle, and the longitudinal component of the electric field $E(\zeta, z)$ satisfies Gauss's equation

$$\frac{\partial E}{\partial z} + \frac{\partial E}{\partial \zeta} = -\frac{e}{\epsilon_0}\int dv\ f(\zeta, v; z). \tag{4}$$

To approximately solve the system (3)-(4), we divide the distribution function into the smooth background term $f_0(v)$ and the perturbation $\hat{f}(\zeta, v; z)$:

$$f = f_0(v) + \hat{f}(\zeta, v; z) \equiv n_0 g(v) + \hat{f}(\zeta, v; z), \tag{5}$$

where the momentum distribution function g associated with f_0 is normalized such that $\int dv\ g(v) = 1$. We introduce the Laplace transform in z and Fourier transform in ζ as

$$\hat{f}_{\omega,k}(v) = \frac{1}{2\pi}\int_0^\infty dz\ e^{i\omega z}\int_{-\infty}^{\infty} d\zeta\ e^{-ik\zeta}\hat{f}(\zeta, v; z) \tag{6}$$

$$E_{\omega,k} = \frac{1}{2\pi}\int_0^\infty dz\ e^{i\omega z}\int_{-\infty}^{\infty} d\zeta\ e^{-ik\zeta}E(\zeta; z); \tag{7}$$

note that the "frequency" ω has the dimensions of inverse length. We treat f_0 and $\hat{f}$ as, respectively, zeroth order and first order quantities, which implies that electric field is also first order $|E| \sim |\hat{f}|$ for finite wavelengths such that $k \neq 0$. We additionally assume that $\omega \ll k$, in which case we obtain the following linear system of equations:

$$(\omega - kv)\hat{f}_{\omega,k} + \frac{ien_0 g'(v)}{mc^2\beta_0\gamma_0^3}E_{\omega,k} = i\hat{f}_k(v; 0) \tag{8}$$

$$-kE_{\omega,k} = \frac{ie}{\epsilon_0}\int dv\ \hat{f}_{\omega,k}(v), \tag{9}$$

where $g'(v) \equiv dg/dv$. Solving (8)-(9) for the electric field in terms of the initial conditions is trivial. We take the distribution function to be of the Klimontovich form

$$f(\zeta, v; z) = \frac{1}{A}\sum_{j=1}^{N_e}\delta[\zeta - \zeta_j(z)]\delta[v - v_j(z)]. \tag{10}$$

Using the initial value $f(\zeta, v; 0)$, the electric field is

$$E_{\omega,k} = \frac{e}{2\pi k\epsilon_0\epsilon(k,\omega)}\frac{1}{A}\sum_{j=1}^{N}\frac{e^{-ik\zeta_j(0)}}{\omega - kv_j(0)}, \tag{11}$$

where we have introduced the normalized dielectric response function

$$\epsilon(k,\omega) = 1 + \frac{\Omega_p^2}{k}\int dv\ \frac{g'(v)}{\omega - kv(0)}. \tag{12}$$

Here $\Omega_p = \sqrt{e^2 n_0/\epsilon_0 mc^2\beta_0\gamma_0^3}$ is the relativistic electron beam plasma frequency in the laboratory frame. The beam bunching factor $b_{k,\omega} = (kA\epsilon_0/eN)E_{\omega,k}$. Taking the inverse Laplace transform, our method yields the solution

$$b_k(z) = \frac{i}{2\pi N}\int_L d\omega\ \frac{e^{i\omega z}}{\varepsilon(k,\omega)}\sum_{j=1}^{N}\frac{e^{-ik\zeta_j(0)}}{\omega - kv_j(0)}, \tag{13}$$

where the integral along the Landau contour L can be performed by finding the poles and evaluating the residues. We will find it instructive to separate the poles into two groups: the first obtained from the zeros of the dielectric function defined by $\omega_q : \varepsilon(k, \omega_q) = 0$; the second are given by $\omega = kv_j(0)$ for $j = 1, 2, \ldots, N$. We use the superscript C to distinguish the part of the bunching factor arising from the former set of poles with $\omega = \omega_q$, while we identify the latter poles the superscript I:

$$b_k(z) = b_k^C(z) + b_k^I(z) \tag{14}$$

with

$$b_k^C(z) = \sum_q e^{-i\omega_q}\frac{1}{\varepsilon'(k,\omega_q)}\frac{1}{N}\sum_{j=1}^{N}\frac{e^{-ik\zeta_j(0)}}{\omega_q - kv_j(0)} \tag{15}$$

$$b_k^I(z) = \frac{1}{N_e}\sum_{j=1}^{N}\frac{e^{-ik(\zeta_j^0 + v_j(0))z}}{\varepsilon[k, kv_j(0)]}. \tag{16}$$

The modes in Eq. (15) oscillate at the frequencies ω_q, and therefore represent the collective motion associated with the plasma wave dynamics. On the other hand, the individual part of the bunching $b_k^I(z)$ involves a sum over the

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independent particle motion. This decomposition is a precise formulation of that first introduced by Pines and Bohm [8]. The relative importance is determined by comparing the density perturbation length scale $\sim 1/k$ with the Debye length $\lambda_D = \sigma_v/\Omega_p$.

To make these statements explicit, we assume that the smooth part of the momentum distribution is a Gaussian,

$$g(v) = \exp(-v^2/2\sigma_v^2)/\sqrt{2\pi}\sigma_v. \quad (17)$$

First, we show that $b_k^C(z)$ yields the familiar plasma wave dynamics in the limit $k\lambda_D \ll 1$. In this case, the diectric function is approximately given by $\epsilon(k,\omega) \approx 1 - \Omega_p^2/\omega^2$, and there are two poles whose magnitude equals plasma frequency $\omega_q = \pm\Omega_p$ [9]. When $k\lambda_D \ll 1$ we also have $\Omega_p \gg kv_j(0)$ for most values of $v_j(0)$, and collective part of the bunching Eq. (15) can be approximately written as

$$\begin{aligned} b_k^C(z) \approx \cos(\Omega_p z)\frac{1}{N}\sum_{j=1}^{N} e^{-ik\zeta_j(0)} \\ - \frac{ik}{\Omega_p}\sin(\Omega_p z)\frac{1}{N}\sum_{j=1}^{N} v_j(0)e^{-ik\zeta_j(0)}. \end{aligned} \quad (18)$$

A similar expression can be written for the collective velocity bunching $\sum_j v_j(z)e^{-ik\zeta_j(z)}/N$. The two averages describe the plasma wave, and we see that the bunching oscillates between its initial value and that of the collective velocity. For a beam initially described by shot-noise statistics, $b_k(0) \sim 1/\sqrt{N}$, while at $z = 0$ the collective velocity $\sim \sigma_v/\sqrt{N}$. Thus, when $k\lambda_D \ll 1$ the fluctuations in $\langle |b_k|^2 \rangle$ decrease from $1/N$ at $z = 0$ to $(k\lambda_D)^2/N$ after a quarter plasma period. At the same time the fluctuations in the collective velocity increase, and eventually returns to its initial value at $\Omega z = 2\pi$. This effect is well-known in microwave devices, and recently it was proposed that such "shot noise suppression" could be useful for FEL applications [1, 2]. We will show the Debye length for present and proposed devices is too long to be applied at x-ray wavelengths, but it may find use at longer wavelengths; for example, the microbunching instability can severely degrade beam quality during compression, and smoothing out fluctuations at and below optical wavelengths may prove to be a useful manipulation. The collective plasma response was also studied analytically in 3D in [7], with comparable qualitative results. Next, we turn to the individual component, which was not the focus of these other studies.

The individual component can be computed as

$$\left\langle \left|b_k^I(z)\right|^2 \right\rangle = \frac{1}{N}\frac{(k\lambda_D)^2}{1+(k\lambda_D)^2}. \quad (19)$$

This is identical to the equilibrium fluctuations in a classical plasma (2). Thus, the bunching squared is comprised of a collective part that is important for $k\lambda_D \ll 1$ and which depends on the initial statistical distribution of the particles, and an individual component that matches the equilibrium fluctuations and is time stationary. In addition, there are

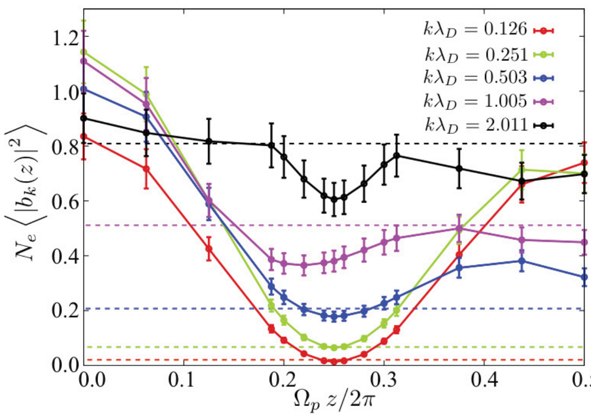

Figure 1: Evolution of the bunching for an electron distribution whose positions are initially random (given by shot noise) and whose velocities are Gaussian distributed. We average $|b_k(z)|^2$ over 100 statistically independent instances to obtain $\langle |b_k(z)|^2 \rangle$. The density fluctuations reach a minimum after one-quarter plasma period, with the value close to that predicted by $\langle |b_k^I|^2 \rangle$ (dotted lines).

cross-terms from the b_k^C and b_k^I ; simulations indicate that they comprise a small correction if $k\lambda_D \ll 1$, while modifying our simple results by $\sim 20\%$ if $k\lambda_D \gtrsim 1$.

To demonstrate the fluctuation evolution from a beam that is initially random and uncorrelated (having shot-noise type statistics), we have modified the galactic simulation code GADGET-2 [10] to simulate a repulsive rather than attractive force. i.e., we use GADGET-2 with "negative gravity." GADGET-2 is a massively parallel code that solves the N-body problem using the fast multipole expansion. By dividing the long-range gravitational (or electrostatic) force into a short range part that is solved exactly and a long-range part that is evaluated using multipole expressions, it reduces the scaling of the number of operations required to solve the N body problem from $O(N^2)$ to $O(N \log N)$. While there are a few electrostatic codes that also use such methods to solve for the full particle interactions, we chose to modify GADGET-2 because it is open source, freely available, and well-optimized for parallel processing. Finally, we note that traditional particle-in-cell (PIC) codes are only appropriate to simulate mean field dynamics; since PIC codes do not completely resolve the interactions of particles within a given cell and are particularly susceptible to error from discrete particle noise, they are not suitable for the present investigation.

We show the results in Fig. 1. The fluctuation level in $N_e\langle |b_k(z)|^2 \rangle$ decreases from its initial value near unity to a value that is fairly well-predicted by the individual average (19), which we plot as dotted lines.

The results of Fig. 1 were obtained by averaging 100 different independent instances, each of which employ 400000 particles in a box with periodic boundary conditions. Thus, these preliminary studies verifying the basic physics neglect any transverse spreading/focusing; we

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Table 1: Numerical example for two modern FEL facilities. The first column lists the existing hard x-ray LCLS parameters, the second has those of the NGLS soft x-ray device, while the final column includes the LCLS injector parameters at 135 MeV.

	LCLS	NGLS	LCLS injector
γ_0	26693	3914	264
Current [A]	3000	500	40
σ_γ	3.5	0.098	0.0053
$\varepsilon_{x,n}$ [mm-mrad]	0.4	0.6	0.4
$2\pi\lambda_D\vert_{\text{min}}$ [nm]	0.3	1.5	10

chose to focus on this simplification because we felt that it would be simplest to perform and interpret, and consider the results of Fig. 1 to represent an idealized situation that will typically be degraded by other effects. In the future we plan to include more more realistic initial conditions that do not assume a periodic system.

Finally, we need to determine what is the relevant Debye length for relativistic electron beams suitable for x-ray free-electron lasers. To determine the rms spread in $v_j \equiv 1 - \beta_0/\beta_j$, we consider the longitudinal velocity $\beta_j = (1 - {x_j'}^2 - 1/\gamma_j^2)^{1/2}$, where x' is the angle the electron makes with the z-axis. Assuming that $1 - \beta_0 \ll 1$, we have

$$v_j \approx \frac{\gamma_j - \gamma_0}{\gamma_0} - \frac{1}{2}{x_j'}^2. \tag{20}$$

Thus, deviations in v are due to spreads in both the energy and the angle. The rms spread appropriate for the Debye length is therefore

$$\sigma_v^2 = \frac{\sigma_\gamma^2}{\gamma_0^2} + \frac{1}{4}\sigma_{x'}^2. \tag{21}$$

Replacing the angular spread with the normalized beam emittance $\varepsilon_{x,n}$ and transverse size σ_x via $\sigma_{x'} = \varepsilon_{x,n}/\gamma_0\sigma_x$, the Debye length can be written in terms of the current I as

$$\lambda_D = \sqrt{\frac{I_A}{2\gamma_0 I}\left(\frac{\sigma_x^2\sigma_\gamma^2}{\gamma_0^2} + \frac{1}{4\sigma_x^2}\varepsilon_{x,n}^4\right)}, \tag{22}$$

where $I_A = 4\pi\epsilon_0 mc^3/e \approx 17$ kA is the Alfvén current. For a beam with given emittance and energy spread, the minimum possible Debye length can be determined by equating the two terms in the parentheses, yielding

$$\lambda_D\big|_{\text{min}} = \frac{1}{\gamma_0}\sqrt{\frac{I_A}{I}\frac{\sigma_\gamma\varepsilon_{x,n}^2}{2}} = \frac{1}{\gamma_0}\sqrt{\frac{I_A}{2ec}\frac{1}{\mathcal{B}_e}}, \tag{23}$$

where $\mathcal{B}_e$ is the electron beam brightness, i.e., the number of particles per phase space volume. Thus, $(\gamma_0\lambda_D|_{\text{min}})^2$ scales inversely with the electron beam brightness, which can only increase during typical beam transport and acceleration.

To put some concrete numbers on the range of Debye lengths one might have for FEL-type electron beams, we include in Table 1 parameters for hard x-ray generation at the Linac Coherent Light Source (LCLS) from [11] and the proposed soft x-ray Next Generation Light Source (NGLS) facility at Berkeley National Lab [12]. We see that at the target hard x-ray wavelengths of 6 to 1.5 Å the LCLS has $k\lambda_D \gtrsim 1$ and the plasma wave is not important; similar conditions obtain for the NGLS at wavelengths less than 3 nm. For all the listed parameters the minimum Debye length is much less than 100 nm, so that reduction in fluctuations can be expected to be observed in the optical part of the spectrum, as has been recently measured in [13].

Finally, note that at the LCLS $\gamma_0\lambda_D|_{\text{min}}$ is about a factor of three larger at high energy (and current) than it is at the injector. This is because the beam energy spread is increased before compression to mitigate the microbunching stability. To make the plasma wave relevant at high energy and current would therefore first require a method of compressing the beam without increasing energy spread. If one could do this, than reducing the fluctuations in $\langle|b_k|^2\rangle$ to $0.1/N$ would also require an increase in LCLS beam brightness by at least a factor of 4.

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