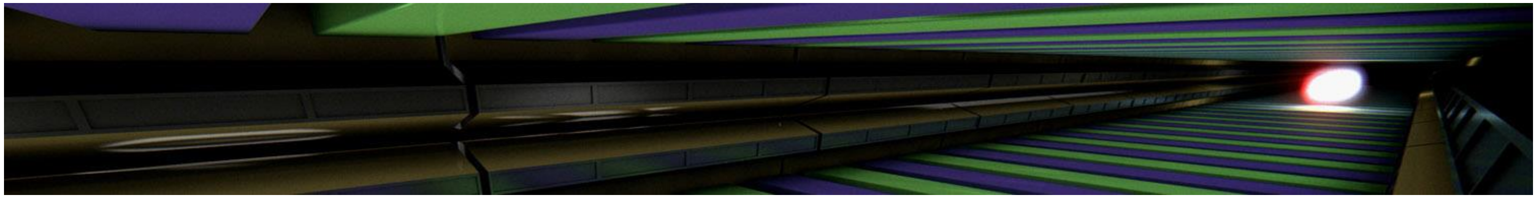

On Quantum Effects in Spontaneous Emission by a Relativistic Electron Beam in an Undulator

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Main Message from this talk:

A drift-diffusion model can be used to describe quantum effects in spontaneous emission by a relativistic electron beam whenever

$$\zeta = \frac{\hbar\omega}{\gamma m c^2} << 1$$

Where ω is the resonance frequency at which the undulator is tuned

Contents:

- Introduction to Fokker-Planck equation
- Applicability parameter / angle-integrated SR spectrum
- Diffusion coefficient / angle-integrated SR spectrum
- Relation with Thomson scattering
- Conclusions

The Fokker-Planck Equation

Define f($\mathcal{E}$,t) the electron distribution function

Define $\Psi(\mathcal{E},\Delta\mathcal{E})\Delta\mathcal{E}$ the Probability to change energy from $\mathcal{E}$ to $\mathcal{E}$-$\Delta\mathcal{E}$ in the time interval Δt

Assume that $\Psi(\mathcal{E},\Delta\mathcal{E})$ only depends on the current value of $\mathcal{E}$, not on the electron history (Markov process)

$$f(\mathcal{E},t+\Delta t)=\int d(\Delta\mathcal{E})\Psi(\mathcal{E}+\Delta\mathcal{E},\Delta\mathcal{E})\,f(\mathcal{E}+\Delta\mathcal{E},t)$$

Fokker-Plank is obtained by Taylor expansion for "small values of $\Delta\mathcal{E}$". We will be interested in examining the **conditions** when such Taylor expansion is acceptable, i.e. to quantify the expression "small values of $\Delta\mathcal{E}$"

Fokker-Planck equation:

$$\frac{\partial}{\partial t} f(\mathcal{E},t) = \frac{\partial}{\partial \mathcal{E}}\left[C_1(\mathcal{E}) f(\mathcal{E},t)\right] + \frac{1}{2}\frac{\partial^2}{\partial \mathcal{E}^2}\left[C_2(\mathcal{E}) f(\mathcal{E},t)\right]$$

$$C_1(\mathcal{E}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int d(\Delta\mathcal{E})\Psi(\mathcal{E},\Delta\mathcal{E})\Delta\mathcal{E}$$

$$C_2(\mathcal{E}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int d(\Delta\mathcal{E})\Psi(\mathcal{E},\Delta\mathcal{E})\Delta\mathcal{E}^2$$

Where C_2 is the diffusion coefficient

How do we find Ψ? **Ψ is given by the physics of the process, in our case SR emission**

Applicability parameter

Physically, $\Psi(\mathcal{E},\Delta\mathcal{E})$ has a very well defined meaning:

$$\frac{1}{\hbar\omega}\frac{dW}{d(\hbar\omega)} = \psi(\mathcal{E},\Delta\mathcal{E})$$

Where $\frac{dW}{d(\hbar\omega)}$ is the photon spectrum.

Furthermore we set

$$\Delta\mathcal{E} = \hbar\omega$$

due to energy conservation

$$\frac{1}{\hbar\omega}\frac{dW}{d(\hbar\omega)} = \psi(\mathcal{E},\Delta\mathcal{E})$$ where $$\frac{dW}{d(\hbar\omega)}$$ is the undulator photon spectrum

Fokker-Plank is obtained by Taylor expansion. **Conditions?** Quantify the expression “small values of $\Delta\mathcal{E}$” with $\Delta\mathcal{E} = \hbar\omega$

Follow this reasoning:

The undulator line ~ $1/N_w$, with N_w number of undulator periods

Then the characteristic spread in electron energy will be equal to $\delta=\gamma mc^2/N_w$

→ Fokker-Planck applicable when

$$\epsilon = \frac{N_w \hbar\omega}{\gamma mc^2} << 1$$

G. Robb and R. Bonifacio, Europhysics Letters 94, 34002 (2011)

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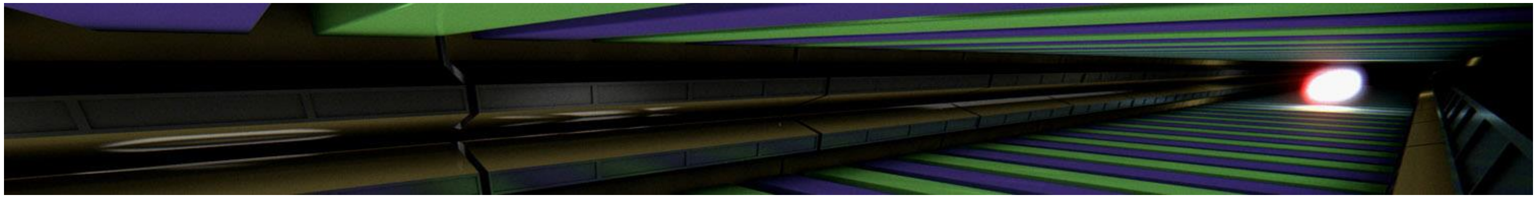
We disagree with this reasoning



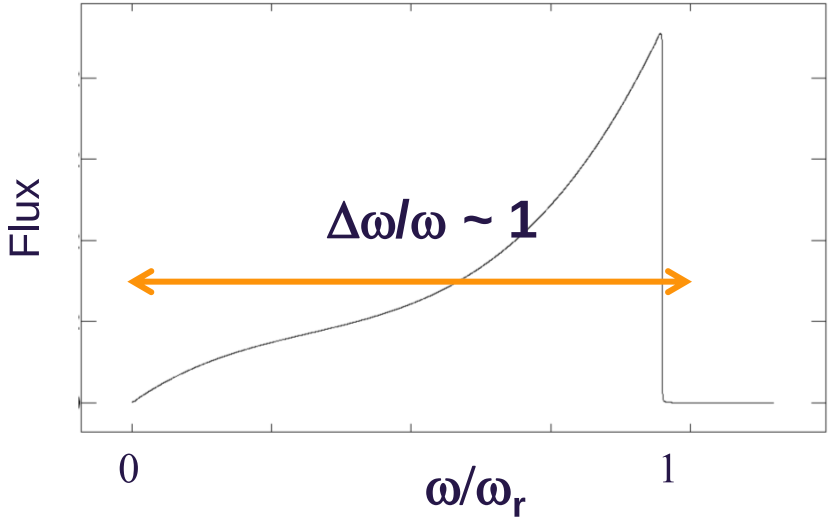
There is a spectral angular dependence in the undulator flux.

The electron feels the recoil from all photons at all angles the angle-integrated spectrum, and the angle-integrated spectrum dos not depend on N_w!

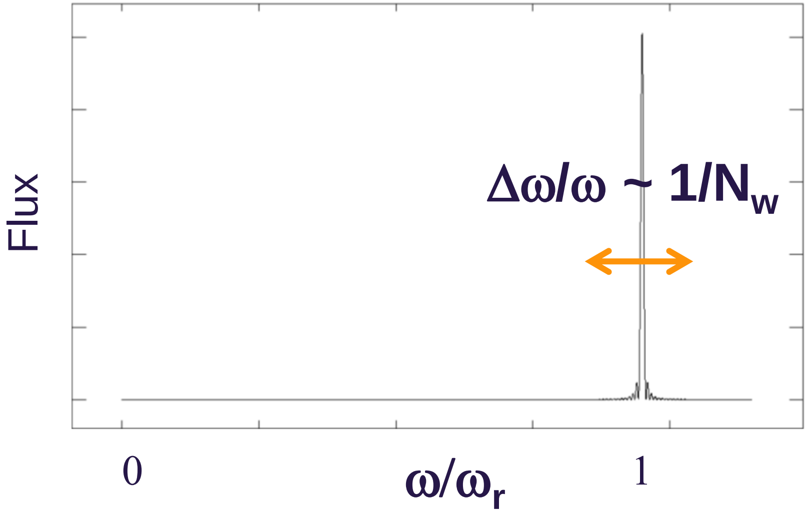
$$\frac{dW}{d(\hbar\omega)}$$ is the angle-integrated photon spectrum

It is true that, on axis, the undulator line ~ $1/N_w$

...but the angle-integrated spectrum does not depend on N_w



Angle-integrated spectrum



Spectrum along one direction

The right parameter is therefore

$$\zeta = \frac{\hbar\omega}{\gamma mc^2} << 1$$

The diffusion coefficient

This fact can be shown analytically in the limiting case for K<<1 and Nw>>1. Using γ^2>>1 (and therefore the paraxial approximation) we have:

$$\frac{dW}{d\omega d\Omega} = \frac{\omega^2 K^2 L_w^2 e^2}{16\pi^2 c^3 \gamma^2} \left\{ \left[1 - \frac{\theta_x^2 \omega}{k_w c} \right]^2 + \left[\frac{\theta_x \theta_y \omega}{k_w c} \right]^2 \right\} \mathrm{sinc}^2 \left[\frac{L_w}{4} \left(C + \frac{\omega \theta^2}{2c} \right) \right]$$

Where θx and θy are the observation angles,

$$C = \omega/(2\bar{\gamma}_z^2 c) - k_w = (\Delta\omega/\omega_{r0}) k_w, \text{ where } \omega = \omega_{r0} + \Delta\omega$$

And the resonance frequency $\omega_{r0} = 2 k_w c \bar{\gamma}_z^2$

We integrate on the solid angle analytically.

By using Nw>>1 we can substitute

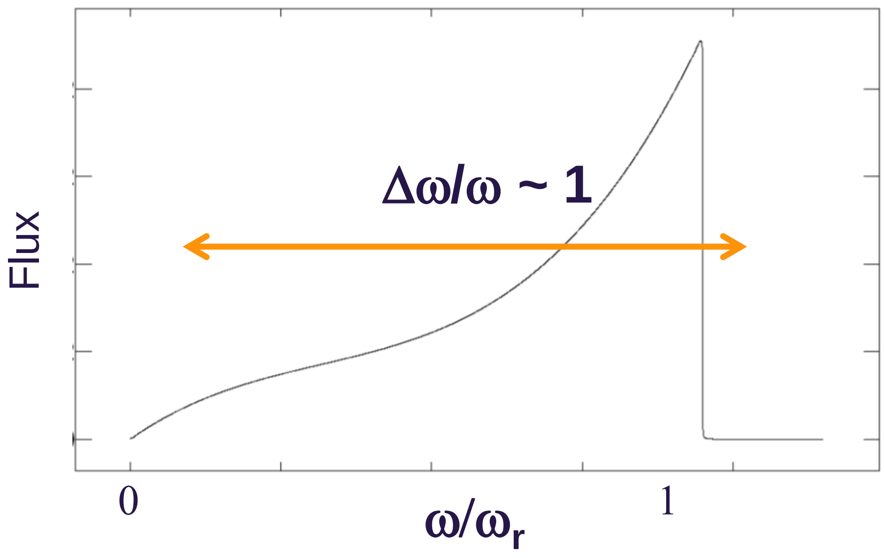
$$\mathrm{sinc}^2[x/a]/(\pi a) \longrightarrow \delta(x) \text{ for } a \longrightarrow 0$$

$$\frac{dW}{d\omega} = \frac{e^2\omega K^2 L_w}{4c^2\gamma^2}\left[1+\left(\frac{\omega}{ck_w\gamma^2}-1\right)^2\right]$$

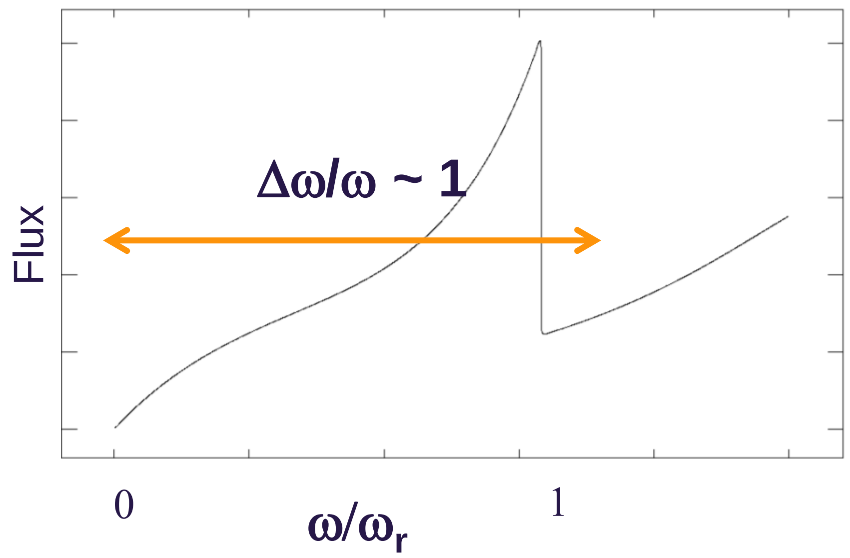
for $\omega < 2c\gamma^2 k_w$, and zero otherwise

This expression is in agreement with many in literature (see e.g. Alferov et al., Sov. Phys. Tech. Phys. 18, 1974)

Conclusion: the linewidth does not depend on N_w



K<<1



K~1

Here we considered the limiting case K<<1, but the same result is valid in general!

(E. Saldin, E. Schneidmiller, M. Yurkov, NIM A 381, 545 (1996))

$$\frac{1}{\hbar\omega}\frac{dW}{d(\hbar\omega)} = \psi(\mathcal{E}, \Delta\mathcal{E})$$

We can now find the quantum diffusion coefficient as

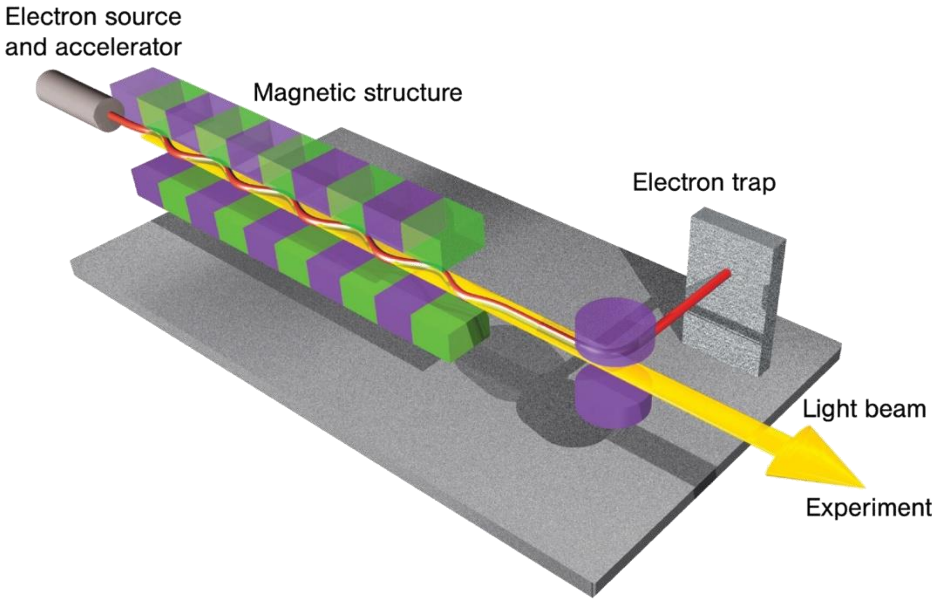
$$\frac{C_2}{m^2c^4} = \frac{d\langle(\Delta\gamma)^2\rangle}{dt} = \frac{c}{L_w}\frac{1}{m^2c^4}\int_0^\infty d\omega\,\hbar\omega\frac{dW}{d\omega} = \frac{7}{15} r_e c \lambda\!\!\!^{-}_c K^2 k_w^3 \gamma^4$$

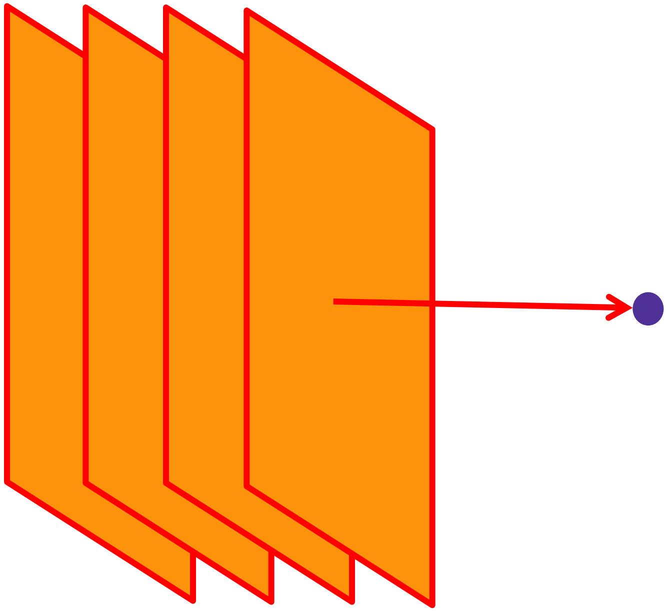
See Ya. S. Derbenev, A. M. Kondratenko and E. L. Saldin, NIMA 193, 415 (1982)

Relation to Thomson scattering

$$K^2 \ll 1 \text{ and } N_w \gg 1$$

Equivalence between





$$\omega_R = \gamma c k_w$$

Undulator radiation

Thomson scattering

Thomson scattering for polarized radiation, rest frame:

$$\frac{d\sigma}{d\Omega_R} = r_e^2 \left[\cos^2(\theta_R)\cos^2(\phi_R) + \sin^2(\phi_R)\right]$$

With S_R the magnitude of the average Poynting vector in the rest frame

$$E_R \simeq \gamma B_L$$
$$B_R = \gamma B_L$$
$$B = \frac{Kmc^2 k_w}{e}$$

$$\Longrightarrow \quad \bar{S}_R = \frac{c}{8\pi}\left(\frac{\gamma K m c^2 k_w}{e}\right)^2$$

The radiation pulse in the rest frame has duration $L_w/(\gamma c)$

$$\frac{dN_{phR}}{d\Omega_R} = \frac{d\sigma}{d\Omega_R}\frac{L_w}{\gamma c}\frac{1}{\hbar\omega_R}\bar{S}_R$$

Using

$$\omega_L(\theta_R) = \gamma\omega_R(1 + \cos\theta_R)$$

The rate of change in the spread in the electron energy change can be found as

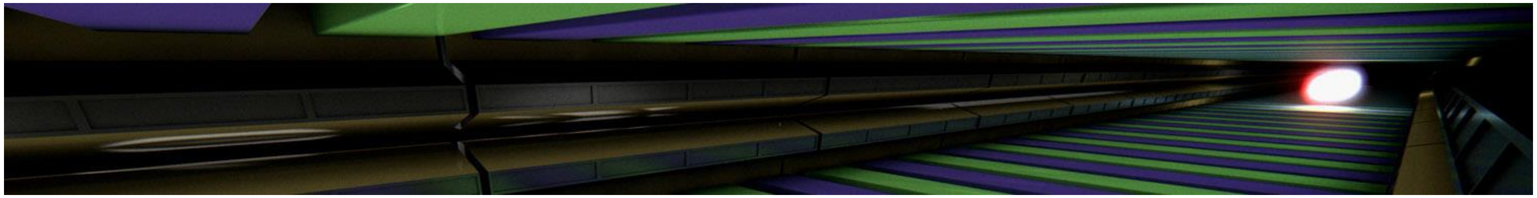
$$\frac{d\langle(\Delta\gamma)^2\rangle}{dt} = \frac{c}{L_w}\int d\Omega_R \left(\frac{\hbar\omega_L(\theta_R)}{mc^2}\right)^2 \frac{dN_{phR}}{d\Omega_R} \Rightarrow \frac{d\langle(\Delta\gamma)^2\rangle}{dt} = \frac{7}{15} r_e c \bar{\lambda}_c K^2 k_w^3 \gamma^4$$

Which is just the diffusion coefficient

(see S. Benson and M.J. Madey, NIMA 237, 55 (1985))

Conclusions

- Quantum effects in spontaneous radiation (SR) can be modeled via a drift-diffusion model in all practical cases of interest
- SR radiation must be calculated within a 3D framework: there is a spectral-angular dependence in SR, and electrons react to photons emitted at all angles
- The linewidth of the angle-integrated spectrum does not depend on the number of undulator periods, and our main conclusion holds



Thank you!