Free-Electron Laser Growing Modes and their Bandwidth

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SELF-CONSISTENT MAXWELL-VLASOV EQUATION

SELF-CONSISTENT MAXWELL-VLASOV EQUATION[†,‡]:

$$\tilde{E} = \int d\hat{P} \, \frac{d\hat{F}}{d\hat{P}} \int_0^{\hat{z}} d\hat{z}' \; e^{i(-\hat{\Delta}_{3D}+\hat{P})} \left\{ \tilde{E} + i\hat{\Lambda}_p^2 \frac{d\tilde{E}}{d\hat{z}'} \right\}$$

YIELDS SOLUTIONS OF THE FORM:

$$\tilde{E} = \sum_n \tilde{V}_n \; e^{s_n \hat{z}}$$

WITH THE DISPERSION RELATION

$$s = \left(1 + is\hat{\Lambda}_p^2\right) D(s) \qquad D(s) = \int d\hat{P} \frac{d\hat{F}}{d\hat{P}} \frac{1}{s + i\left(\hat{P} - \hat{\Delta}_{3D}\right)}$$

† E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, NIM A **313**, 555 (1992).
‡ S. D. Webb, V. N. Litvinenko, and G. Wang, PR ST-AB **14**, 051003 (2011).

FREE-ELECTRON LASER DISPERSION RELATION

PROPERTIES OF THE DISPERSION RELATION

- κ-n distributions yield n+2 order polynomials[†]
- Gaussian yields infinite number of solutions[‡]
- A question: How many growing solutions exist for a given energy distribution?

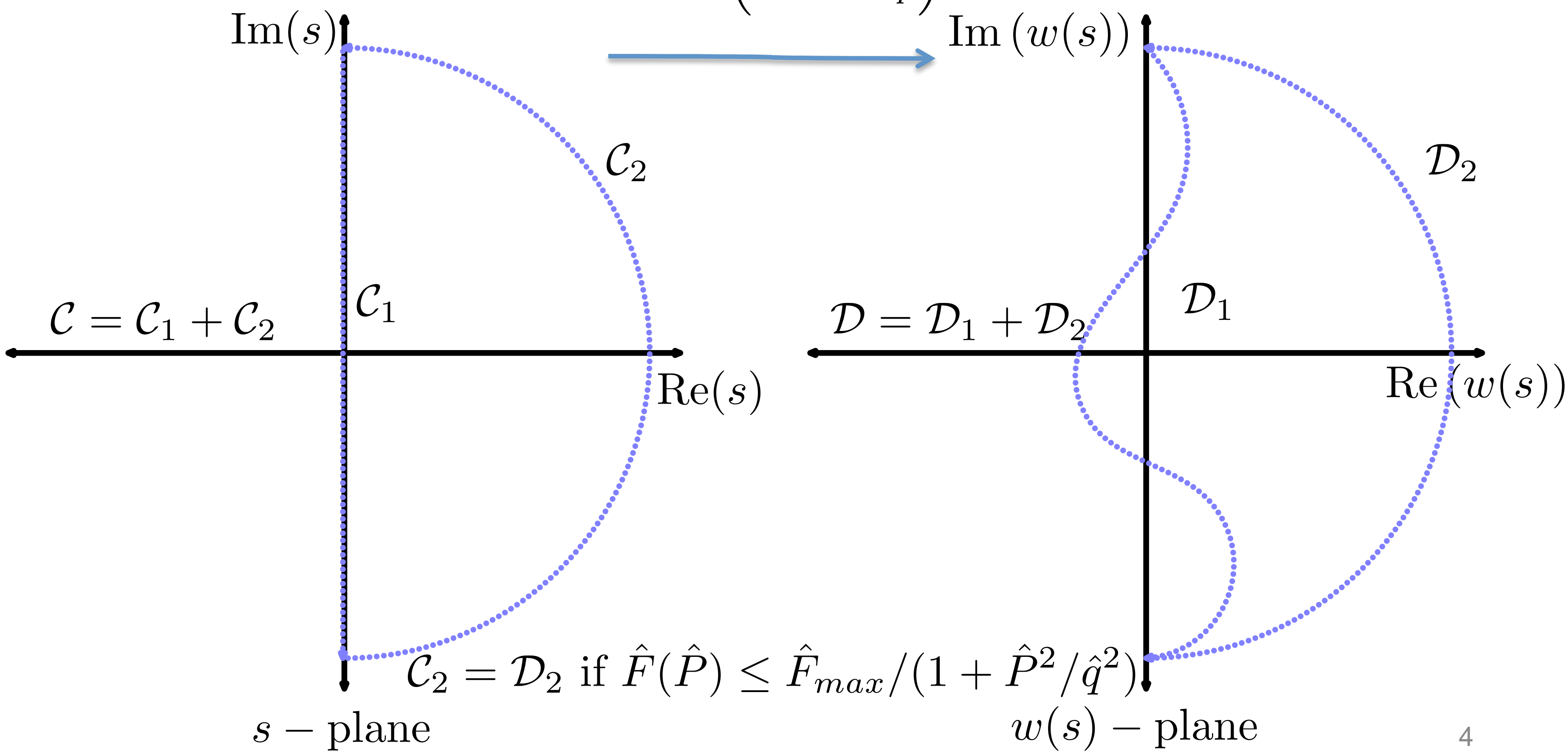
PHYSICS SHOULD NOT DEPEND ON EXACT FUNCTIONAL FORM OF THE ENERGY DISTRIBUTION

† S. D. Webb and V. N. Litvinenko, *Proc. of FEL 2010* MOPB03, 56-59
‡ E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, NIM A **313**, 555 (1992).

Counting the Growing Modes

Mappings & Winding Numbers

$$w(s) = s - \left(1 + is\hat{\Lambda}_p^2\right) D(s)$$



Counting the Growing Modes

Parameterization of Contours

The number of growing modes is equal to the winding number

The contour at infinity is an identity map, so the winding number is determined by the zeros of the map of the vertical contour:

$$w(t) = i \left\{ t + \left(1 - t\hat{\Lambda}_p^2\right) \mathrm{P.V.} \int_{-\infty}^{\infty} d\hat{P} \frac{d\hat{F}}{d\hat{P}} \frac{1}{t - \hat{\Delta}_{3D} + \hat{P}} \right\}$$

$$\underbrace{- \pi \left(1 - t\hat{\Lambda}_p^2\right) \frac{d\hat{F}}{d\hat{P}} \Bigg]_{\hat{P} = \hat{\Delta}_{3D} - t}}_{\text{Crosses imaginary axis at the zeros}}$$

Crosses imaginary axis at the zeros

THE GROWING MODES

WINDING NUMBER & GENERAL RESULT

Contour intersects the imaginary axis when

$$\left(1 - t\hat{\Lambda}_p^2\right) \frac{d\hat{F}}{d\hat{P}}\bigg|_{\hat{\Delta}_{3D}-t} = 0$$

The maximum number of zeros in the right half-plane

$$Z_{max} = (N_{le} + 1)/2$$

THE NUMBER OF GROWING MODES IS EQUAL TO THE NUMBER OF MAXIMA IN THE ENERGY DISTRIBUTION, WITH CUTOFFS DUE TO SPACE CHARGE AND ENERGY SPREAD

THE GROWING MODES

A HIGH-FREQUENCY CUT-OFF

For a single-peaked distribution, we can go further

Maximum cutoff frequency above which there is no amplification

$$\hat{\Delta}^*_{3D} = -\frac{P.V. \int_{-\infty}^{\infty} d\hat{P} \ \frac{1}{\hat{P}} \frac{d\hat{F}}{d\hat{P}}}{1 - \hat{\Lambda}_p^2 \ P.V. \int_{-\infty}^{\infty} d\hat{P} \ \frac{1}{\hat{P}} \frac{d\hat{F}}{d\hat{P}}}$$

Examples:

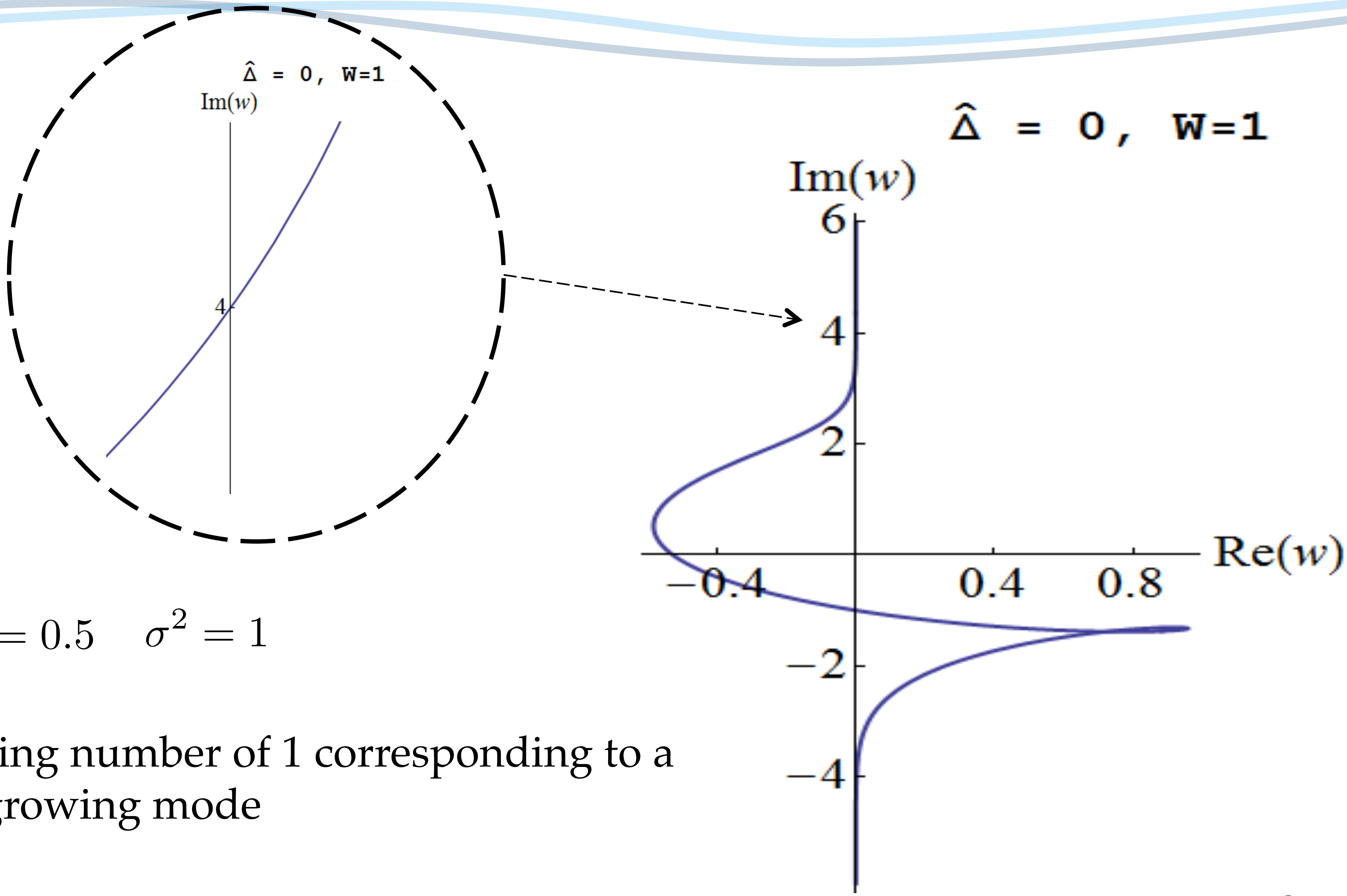
Gaussian:

$$\hat{\Delta}^*_{3D} = \frac{1}{\hat{\sigma}^2 + \hat{\Lambda}_p^2}$$

kappa-n:

$$\hat{\Delta}^*_{3D} = \frac{1}{\frac{2n}{2n-1}\hat{\sigma}^2 + \hat{\Lambda}_p^2}$$

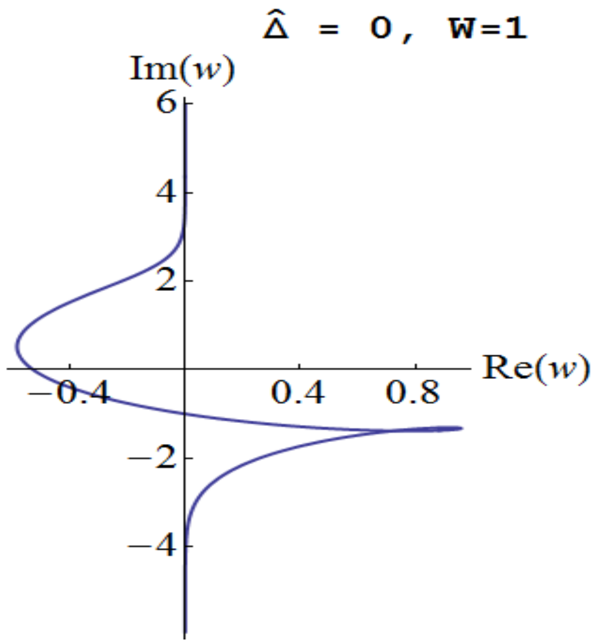
GAUSSIAN ENERGY DISTRIBUTION

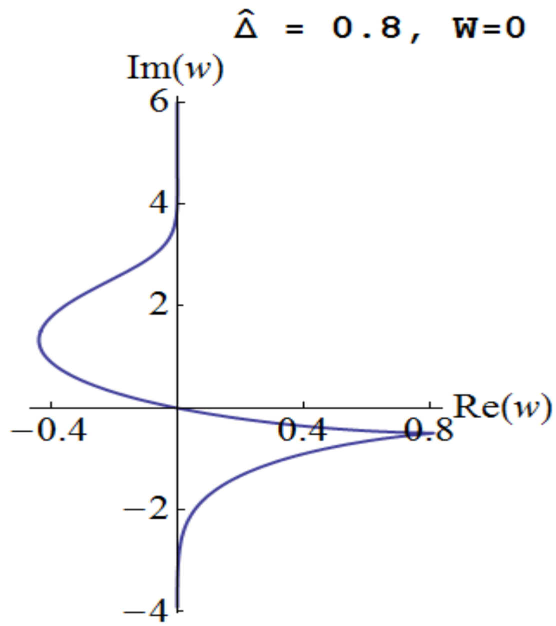


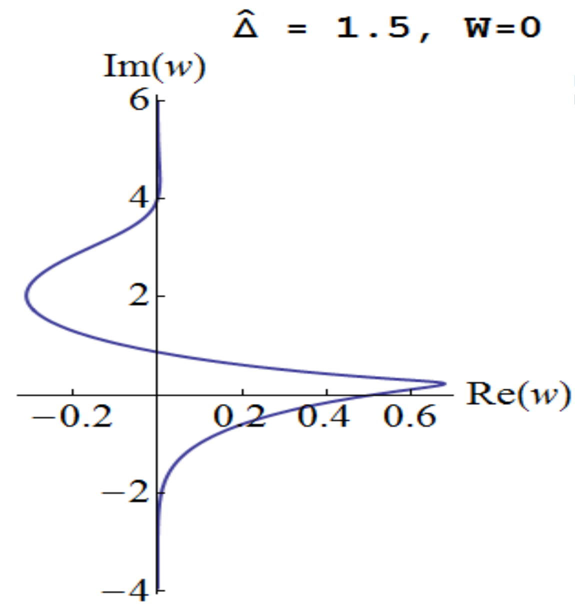
$\hat{\Lambda}_p = 0.5 \quad \sigma^2 = 1$

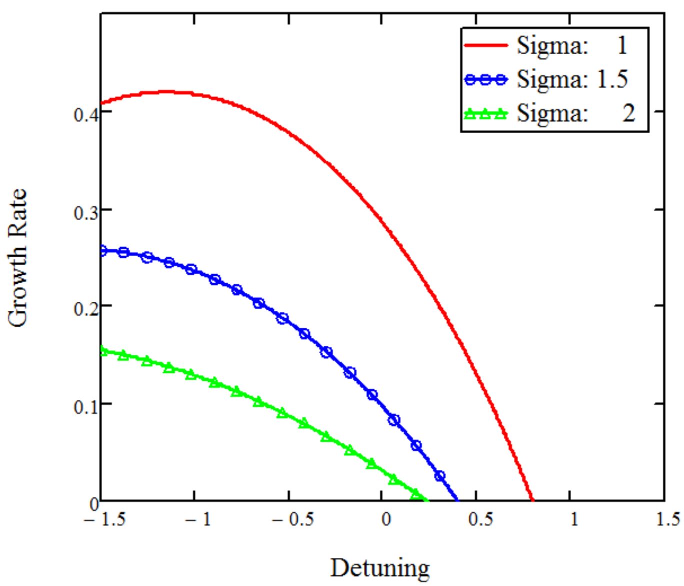
A winding number of 1 corresponding to a single growing mode

Gaussian Energy Distribution







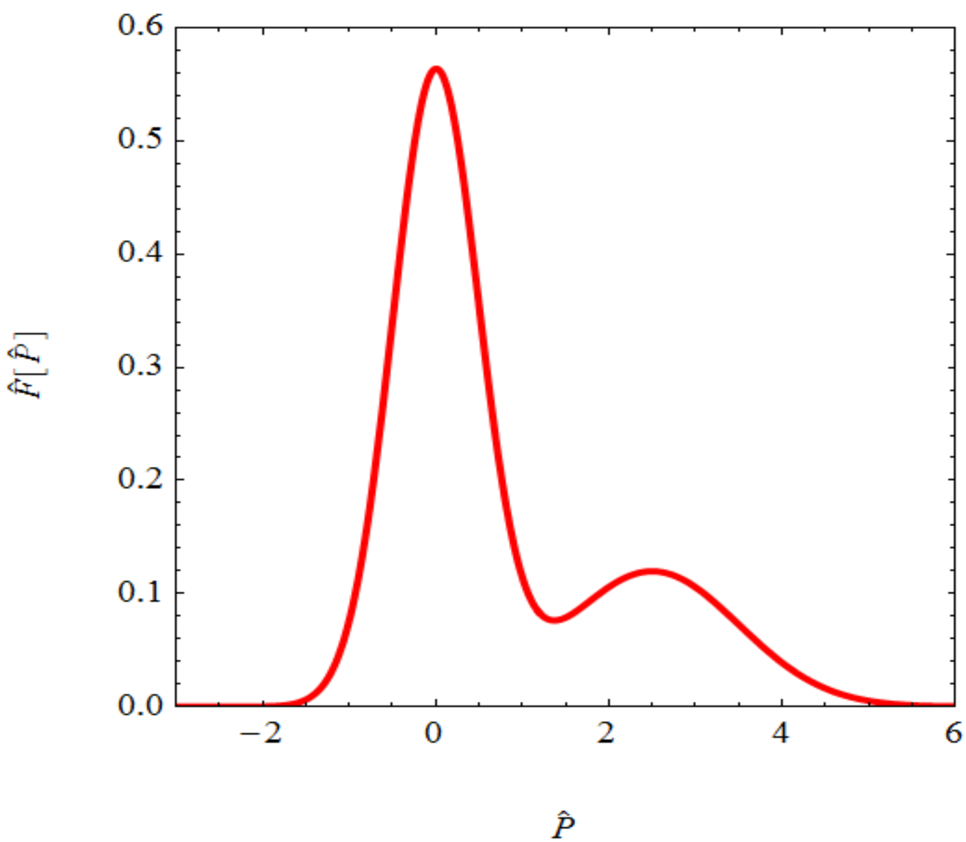


Numerical solution of the dispersion relation:

- One growing roots
- Cutoff detuning agrees with theory
- Winding number =
  Number of growing roots

Double-Gaussian Energy Distribution

$$\hat{F}\left(\hat{P}\right) = \frac{A}{\sqrt{2\pi\sigma_1^2}} \exp\left[-\frac{\hat{P}^2}{2\sigma_1^2}\right] + \frac{1-A}{\sqrt{2\pi\sigma_2^2}} \exp\left[-\frac{(\hat{P}-\hat{P}_2)^2}{2\sigma_2^2}\right]$$

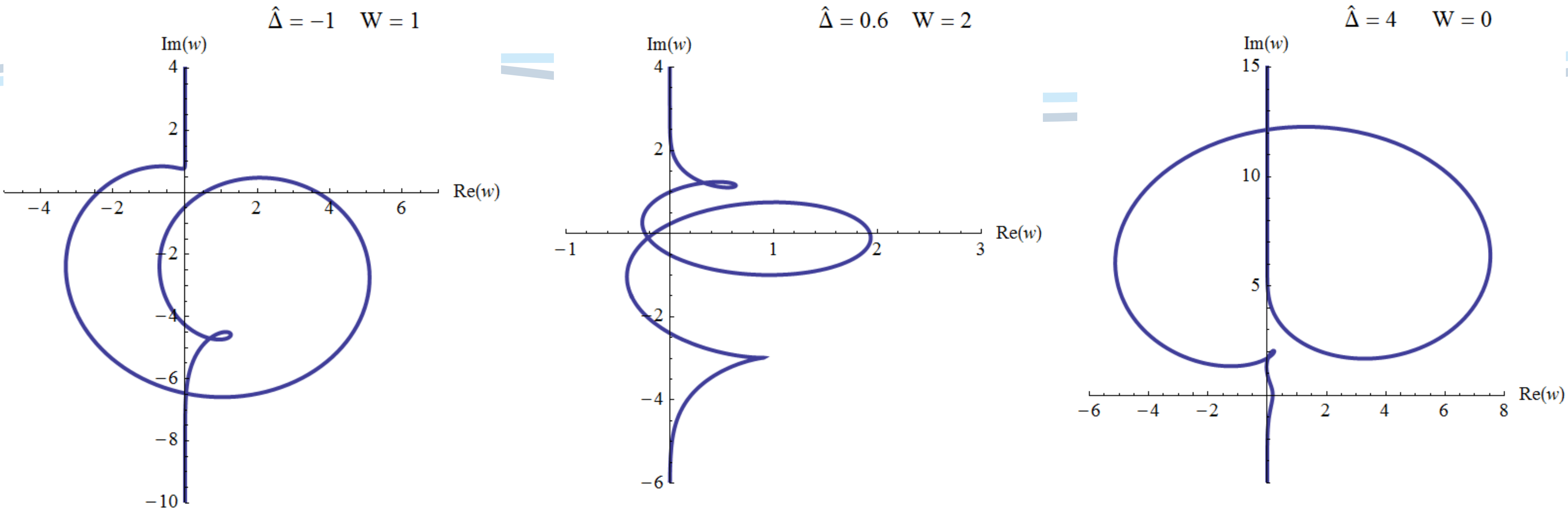


Parameters:

$$\sigma_1 = 0.5$$
$$\sigma_2 = 1.0$$
$$A = 0.7$$
$$\hat{P}_2 = 2.5$$

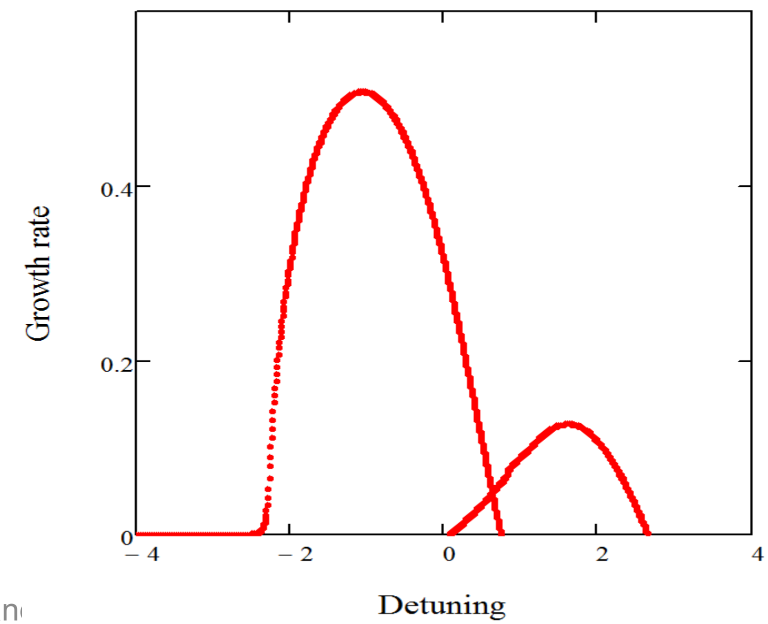
Expect two growing modes from theoretical arguments

Double-Gaussian Energy Distribution



Numerical solution of the dispersion relation:

- Two total growing modes
- Winding number corresponds to growing modes
- Contours have no similarities at different detuning (compare to zero space charge limit)



CONCLUSION

- For an energy distribution with N maxima, there are N total growing modes
- Exact analytical expression exists for an upper frequency cutoff
- Effects should be visible in output power spectrum

Extra Slides

Properties of the Mapping

If the energy distribution is smaller than a Lorentzian everywhere:

$$\hat{F}\left(\hat{P}\right) \leq \frac{\hat{F}_{max}}{1+\hat{P}^2/\hat{q}^2}$$

then the limit vanishes and the arcs at infinity are identical

$$\lim_{|s|\to\infty}\left|\left(1+is\hat{\Lambda}_p^2\right)D(s)\right| = 0$$

The vertical contour is parameterized by

$$w(t) = i\left[t+\left(1-t\hat{\Lambda}_p^2\right)P.V.\int_{-\infty}^{\infty}d\hat{P}\,\frac{d\hat{F}}{d\hat{P}}\frac{1}{t-\hat{\Delta}_{3D}+\hat{P}}\right]-\pi\left(1-t\hat{\Lambda}_p^2\right)\left.\frac{d\hat{F}}{d\hat{P}}\right|_{\hat{P}=\hat{\Delta}_{3D}-t}$$

- Intersects imaginary axis at $w(t)=i\hat{\Lambda}_p^{-2}$
- In the right half-plane for $t=\pm\infty$

NO POLES IN THE RIGHT HALF-PLANE

Dispersion relation poles are dispersion integral poles

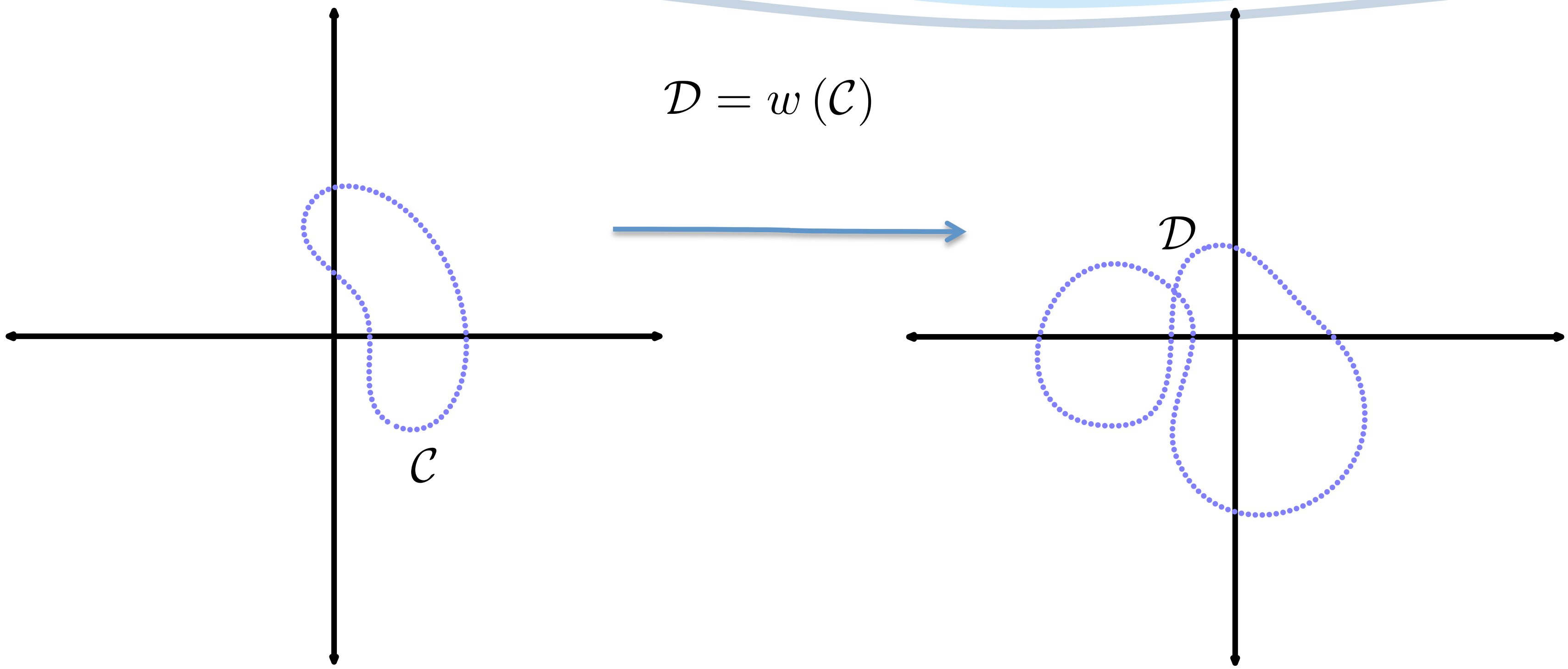
$$w(s) = \underbrace{s - \left(1 + is\hat{\Lambda}_p^2\right)}_{\text{Analytic}} \times \underbrace{D(s)}_{\text{Poles}}$$

The derivative in the right half-plane is finite

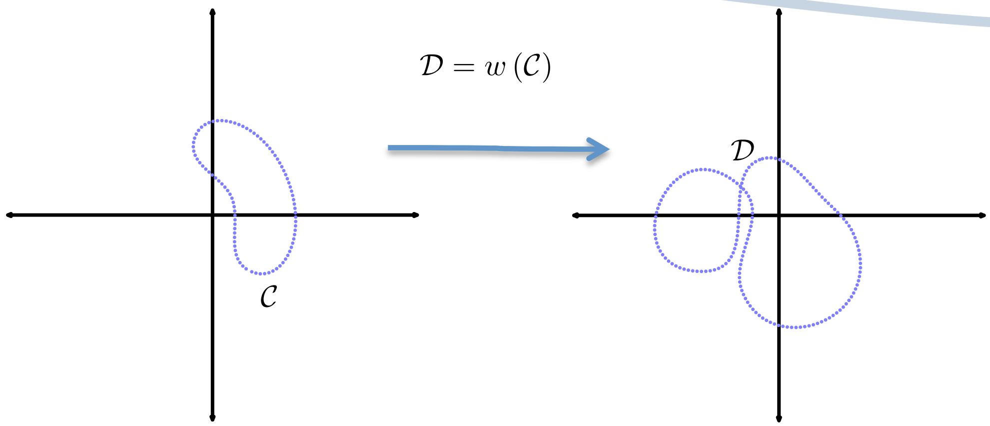
$$\left|\frac{d}{ds}D(s)\right| = \left|\int_{-\infty}^{\infty} d\hat{P}\ \frac{d\hat{F}}{d\hat{P}} \frac{1}{\left[s + i(\hat{P} - \hat{\Delta}_{3D})\right]^2}\right| \leq \frac{1}{\mathrm{Re}(s)^2} \int_{-\infty}^{\infty} d\hat{P} \left|\frac{d\hat{F}}{d\hat{P}}\right|$$

Thus, the dispersion integral is analytic in the right half-plane

ARGUMENT PRINCIPLE OVERVIEW



Argument Principle Overview



- No zeros or poles on the contour
- Function is meromorphic inside the contour

Argument Principle -- *The change in the complex phase of w along the contour C is related to the number of zeros and poles inside C by:*

$$\frac{1}{2\pi}\Delta_{\mathcal{C}} \arg w(z) = Z - P$$

with positive (counterclockwise) orientation.