EFFECTS OF QUANTUM DIFFUSION ON ELECTRON TRAJECTORIES AND SPONTANEOUS RADIATION EMISSION.

I. Agapov*, G. A. Geloni, European XFEL GmbH, 22761 Hamburg, Germany
O. Chubar, Brookhaven National Laboratory, Upton NY 11973, USA

Abstract

For various cases, e.g. in the long undulator sections of the European XFEL, quantum diffusion and energy loss have a noticable effect on the electron trajectory, which in turn effects the properties of the emitted radiation. We discuss approaches to modelling the electron dynamics taking this into account and the effect it has on spontaneous radiation emission.

INTRODUCTION

When calculating synchrotron radiation emitted in long undulator sections, beam energy distribution dilution caused by quantum fluctuations in the radiated energy has to be taken into account. At the same time, one often does not need to consider the quantum nature of the radiation itself, so it is sufficient to consider radiation computed classically, averaged over an ensemble of fluctuating trajectories. In this note the approaches for such random trajectory calculations are discussed and various numerical examples given.

DIFFUSION APPROXIMATION

The equations of motion in a magnetic field without taking energy change into account are

$$\dot{\boldsymbol{\beta}} = \frac{e}{m_e \gamma} \boldsymbol{\beta} \times \mathbf{B} \tag{1}$$

Now assume that in time Δt the electron looses energy due to (random) emission of radiation quanta. The approach to including the energy loss is to consider an electron moving on a certain trajectory where the emitted spectrum is known, such as a circular path or a sinusoidal path in an undulator, and then calculating its mean and variance. For an electron moving on a path with a constant radius of curvature ρ this can be obtained according to [1]. The assumption is that during the time $\Delta t = \rho/\gamma c$ the radius of curvature stays constant (the time needed for the radiation cone to sweep by an angle $1/\gamma$). Since $\gamma m_e [GeV/c] = \rho B[T] \cdot 0.2998$ the requirement that the path length during this 'emission time' is much less than the undulator period

$$\Delta t \cdot c = \rho/\gamma = m_e[GeV]/0.2998 B[T] \ll l_w[m]$$

or roughly

$$1/B[T] \ll 6 \cdot l_w[cm]$$

* ilya.agapov@xfel.eu

This condition is not always fulfilled for XFEL undulators when K is small. Consider an electron moving along a circular trajectory with constant curvature. The relevant radiation characteristics are summarized below.

Table 1: Summary of SR properties

Power spectrum	$\mathcal{P}(\omega,t) = \frac{P_\gamma}{\omega_c} S(\frac{\omega}{\omega_c})$
Photon distribution	$n(u) = \frac{P_\gamma}{u_c^2} F(\frac{u}{u_c}) \quad u = \hbar\omega$
Universal SR functions	$S(\xi) = \frac{9\sqrt{3}}{8\pi}\xi \int_\xi^\infty K_{5/3}(\hat{\xi}) d\hat{\xi}$
	$F(\xi) = \frac{1}{\xi} S(\xi)$
Critical energy	$\omega_c = \frac{c\gamma^3}{\rho} \quad u_c = \frac{3}{2}\frac{\hbar c \gamma^3}{\rho}$
Total power	$P_\gamma = \frac{c C_\gamma}{2\pi}\frac{E^4}{\rho^2}$
	$C_\gamma = \frac{4\pi}{3}\frac{r_e}{(m_e c^2)^3}$
Curvature	$\frac{1}{\rho}\left[m^{-1}\right] \approx 0.2998 \frac{B[T]}{E[GeV]}$
Photon statistics	$\langle N \rangle = \frac{15\sqrt{3}}{8}\frac{P_\gamma}{u_c} \quad \langle u \rangle = \frac{8}{15\sqrt{3}} u_c$
	$\langle u^2 \rangle = \frac{11}{27} u_c^2$

In the diffusion approximation the energy change can be described by a stochastic differential equation

$$\frac{d\gamma}{dt} = a(\gamma) + b^2(\gamma) dW_t \tag{2}$$

The drift coefficient $a(\gamma)$ is describing the continuous loss of energy, and the diffusion coefficient $b(\gamma)$ the fluctuations.

$$m_e c^2 a(\gamma) = \lim_{\Delta t \to 0} \frac{\langle \Delta E(\Delta t) \rangle}{\Delta t}$$

$$\left(m_e c^2\right)^2 b^2(\gamma) = \lim_{\Delta t \to 0} \frac{\langle \Delta E^2(\Delta t) \rangle - \langle \Delta E(\Delta t) \rangle^2}{\Delta t}$$

where $E(\Delta t)$ is the energy loss during time Δt. Assuming that the energy of each emitted photon and number of photons emitted are not correlated, and the latter has Poisson statistics with the property $\langle N^2 \rangle = \langle N \rangle^2 + \langle N \rangle$, one has

$$\frac{\langle \Delta E(\Delta t) \rangle}{\Delta t} \to \langle N u \rangle = \langle N \rangle \langle u \rangle = P_\gamma$$

(which is exactly the mean power loss definition), and

$$\frac{\langle \Delta E^2 \rangle}{\Delta t} = \frac{\langle N^2(\Delta t) u^2 \rangle}{\Delta t} = \frac{\langle N^2(\Delta t) \rangle \langle u^2 \rangle}{\Delta t} = \frac{\left(\langle N \rangle \Delta t + \langle N \rangle^2 \Delta t^2\right) \langle u^2 \rangle}{\Delta t} \to \langle N \rangle \langle u^2 \rangle$$

and furthermore observing that

$$\lim_{\Delta t \to 0} \frac{\langle \Delta E \rangle^2}{\Delta t} = \lim_{\Delta t \to 0} P_\gamma^2 \Delta t = 0$$

we arrive at

$$\begin{aligned} a(\gamma) &= \frac{P_\gamma}{m_e c^2} \\ b^2(\gamma) &= \frac{55\sqrt{3}}{72 m_e^2 c^4} P_\gamma u_c \end{aligned} \tag{3}$$

Noting that in the planar undulator the mean field is

$$B = \frac{\pi K m_e}{l_w c}$$

and using expressions from Table 1 we get

$$\frac{\langle d(\delta\gamma)^2 \rangle}{d(ct)} = \frac{b^2(\gamma)}{c} = \frac{\gamma^4}{l_w^3} K^3 \times g$$

Where g is some numerical constant. Following similar approach, but taking the undulator radiation spectrum as a starting point, Saldin et. al. [2] derived the following approximate expression for diffusion in an ideal undulator field

$$\frac{\langle d(\delta\gamma)^2 \rangle}{d(ct)} = \frac{122\pi^3}{15} \lambda_c r_e \frac{\gamma^4}{l_w^3} K^3 \times f(K)$$

$$f(K) = \begin{cases} 1.20 + \dfrac{1}{K + 1.50K^2 + 0.95K^3}, & \text{planar} \\ 1.42 + \dfrac{1}{K + 1.33K^2 + 0.40K^3}, & \text{helical} \end{cases}$$

For $K << 1$ the two approaches lead to different asymptotics, and the approach based on local field value should not be used. For $K > 1$ both approaches can be used. For an XFEL 40mm and 68mm undulators, the diffusion coefficients for different beam energies and K parameters are shown in Fig. 1. Electron trajectories are calculated by solving 1 and 2 Numerically. Since the characteristic time scales in these equations are different, they can be solved with different time step. Furthermore, whereas 1 is oscillatory and for solving it 4th order Runge-Kutta method is preferable, 2 can be solved by a simpler stochastic method, e.g. leap-frog.

MONTE CARLO METHOD

In [3] and references therein a Monte Carlo approach for generating synchrotron radiation spectrum has been described. We now use similar algorithm to compare the diffusion approximation to direct Monte Carlo simulations of the emitted photons taking electron recoil into account. In the following the algorithm is briefly outlined.

First, one assumes that photon emission is a Poisson process with a mean free path ct_{MF}. It is assumed that over

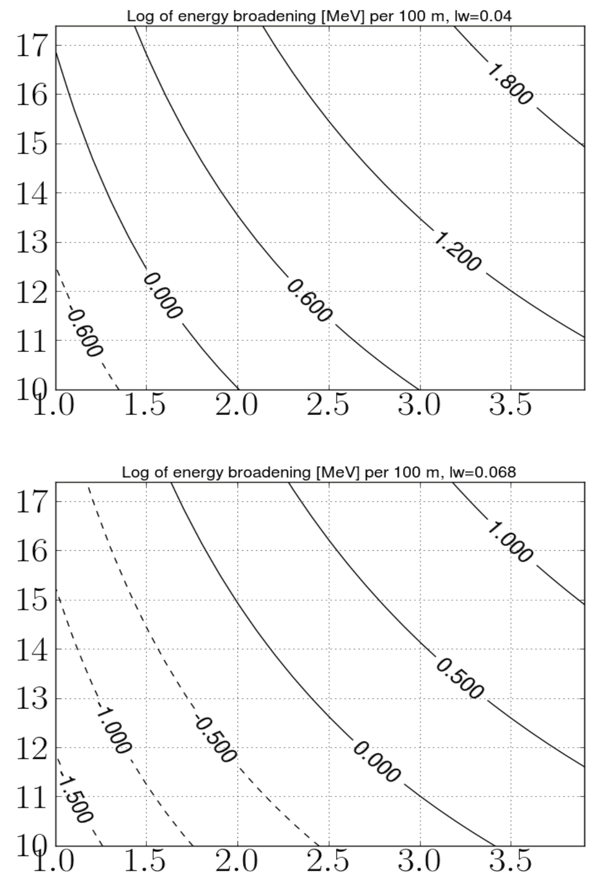

Figure 1: Diffusion coefficients for SASE XFEL undulators. Vertical axis is beam energy, hotizontal is K

this length the electron path is circular (or that the radiation spectrum can be from certain considerations approximated by that of an electron travelling on some average circular path). Then one generates a random time according to Poisson distribution and then simulated electron recoil according to relativistic kinematics

$$\Delta\beta_x = \beta_x - \frac{E_\omega}{m_e c^2 \gamma} \sin \frac{\beta_x}{\beta_z}$$

$$\Delta\beta_z = \beta_z - \frac{E_\omega}{m_e c^2 \gamma} \cos \frac{\beta_x}{\beta_z}$$

$$\Delta\gamma = \gamma - \frac{E_\omega}{m_e c^2}$$

Now the task is to generate photon energies E_ω according to the universal function, which is up to a constant factor

$$F(\xi) = \int_\xi^\infty K_{5/3}(\hat{\xi}) d\hat{\xi}$$

Sampling from such a distribution is accomplished in a standard way by generating a uniformly distributed number y on $[0, 1]$ and then calculating $F_{INT}^{-1}(y)$, where

ISBN 978-3-95450-122-9

$$F_{INT}(y) = (2.19)^{-1} \int_0^y F(\xi) d\xi$$

is the appropriately normalized cumulative distribution function

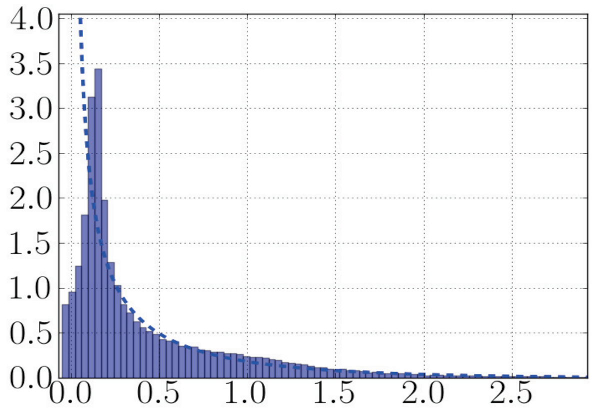

Figure 2: Spectrum of photons generated with the Monte Carlo method (normalized), and the corresponding theoretical universal distribution function

A numerical fit with a 3rd order Chebyshev polynomial leads to the following approximation for $F_{INT}^{-1}(y)$ [1].

$$\begin{array}{ll} -1.87 + 4.31T_1(y) - 1.80T_2(y) + 0.99T_3(y) & y < 0.889 \\ -0.48ln(1-y) & y \geq 0.889 \end{array} \quad (4)$$

Comparison of stochastic integration to direct Monte Carlo simulations show consistent results for the parameter space of the European XFEL (see Fig. 3). For $K << 1$ the direct Monte Carlo method described here will not be legitimate. Although not of direct practical importance, this case might be interesting for calculating inverse Compton scattered photons [4] with the help of SR theory in appropriate rest frame.

REFERENCES

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[2] Saldin et al., "Calculation of energy diffusion in an electron beam due to quantum fluctuations of undulator radiation", NIM A 381.

[3] H. Burkhard., "Monte Carlo generation of the energy spectrum of synchrotron radiation", Eurotev-Report-2007-018.

[4] Petrillo et al., "Phase-Space distribution of electron beam emerging from Compton/Thomson by an intense laser beam", EPL 101 10008, 2013

[1] The author has not been able to reproduce the results from [3] with the fitting coefficients quoted there, and the fitting in this note is different. The accuracy of the fitting formula used in the present work can be of course improved by considering piecewise approximations with more intervals or polynomials of higher order, this however does not look necessary for the purpose considered.

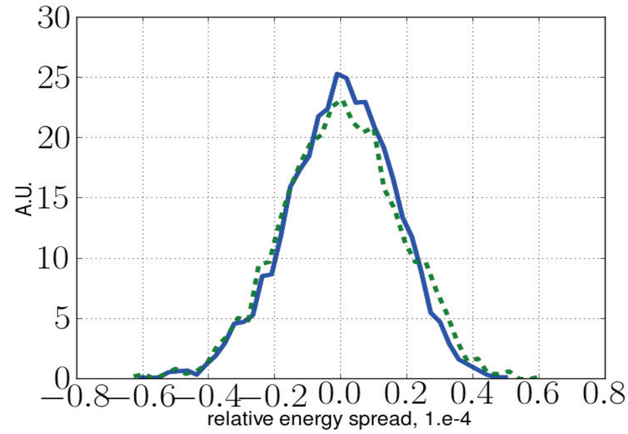

Figure 3: Energy spread γ/γ_0 at the undulator exit, from initially delta-distributed beam. Direct Monte Carlo (solid) and stochastic integration (dashed), for 17.0 GeV Beam, K=1.0, over 100m

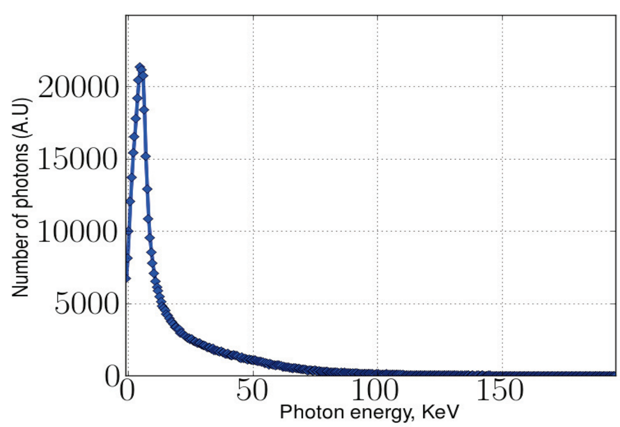

Figure 4: Spectrum of emitted photons E=17.0 GeV, K=1.0

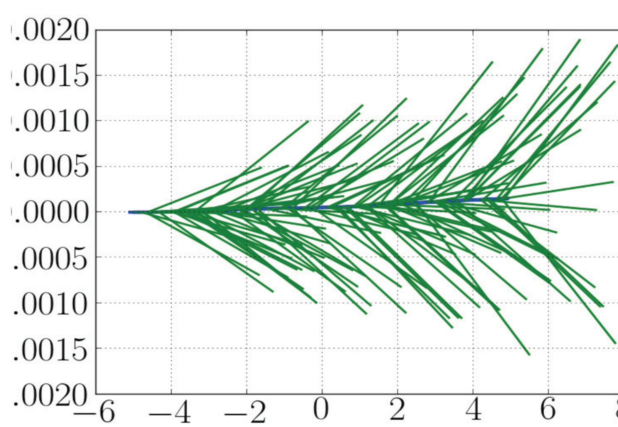

Figure 5: Monte Carlo simulation of emitted photons from one electron, E=17.0 GeV, K=10