EMITTANCE RECONSTRUCTION FROM MEASURED BEAM SIZES*

J. Giner-Navarro, A. Faus-Golfe, J. Fuentes, J. Navarro, J. Resta-Lopez
IFIC (CSIC-UV), Valencia, Spain

Abstract

In this paper we analyze the projected emittance (2D) and the intrinsic emittance (4D) reconstruction method by using the beam size measurements at different locations. We have studied analytically the conditions of solvability of the systems of equations involved in this process and we have obtained some rules about the locations of the measurement stations to avoid unphysical results. Simulations have been made to test the robustness of the algorithm in realistic scenarios with high coupling. The special case of the multi-Optical Transition Radiation system (m-OTR), made of four measurement stations, in the Extraction Line (EXT) of ATF2 is being studied in much detail. The results of these studies will be very useful to better determine the location of the emittance measurement stations in the diagnostic sections of Future Linear Colliders.

INTRODUCTION

The reconstruction of the projected emittance (2D) and the intrinsic emittance (4D) implies the computation of the entire beam matrix envelope [1] at a certain location of a beam line, which can be done from measurements and linear transformations of the beam distribution [2, 3]. These emittances are obtained by numerically solving three separated systems of coupled equations. In this paper we present first an analytical study of the conditions of solvability of these systems of equations and their implications on the emittance reconstruction in order to avoid unphysical results [4]. In a second part we perform realistic tracking simulations in high coupling scenarios in the Extraction Line (EXT) of ATF2 to test the robustness of the method in the m-OTR system of this line.

2D AND 4D EMITTANCE RECONSTRUCTION

The transverse beam phase space could be described by the transverse beam envelope matrix [1]:

$$\sigma = \begin{pmatrix} \langle x^2\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle xx'\rangle & \langle x'^2\rangle & \langle x'y\rangle & \langle x'y'\rangle \\ \langle xy\rangle & \langle x'y\rangle & \langle y^2\rangle & \langle yy'\rangle \\ \langle xy'\rangle & \langle x'y'\rangle & \langle yy'\rangle & \langle y'^2\rangle \end{pmatrix} \tag{1}$$

It consists of a symmetric matrix whose ten independent elements are the second moments of the beam distribution. For instance, $\sigma_1 = \langle x^2\rangle$ and $\sigma_8 = \langle y^2\rangle$ are the horizontal and vertical beam size, respectively. The elements σ_3, σ_4, σ_6 and σ_7 represent the coupling terms.

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The measurement of the second moments $\langle x^2\rangle$, $\langle y^2\rangle$ and $\langle xy\rangle$ at different locations of the beam line allows to reconstruct the ten elements of the transverse beam envelope matrix (1). The horizontal and vertical projected emittance (2D) as well as the intrinsic emittance (4D) could be calculated directly from these terms. Let us denote the measured values by $\hat{\sigma}_1^{(i)}$, $\hat{\sigma}_8^{(i)}$, $\hat{\sigma}_3^{(i)}$ at the measurement stations labelled with $i = 1, 2, \ldots, N$, being N the number of stations. Assuming that the transport matrices are uncoupled, the beam matrix envelope could be determined by solving the three systems of coupled linear equations:

$$M_X \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_5 \end{pmatrix} = \begin{pmatrix} \hat{\sigma}_1^{(1)} \\ \hat{\sigma}_1^{(2)} \\ \ldots \\ \hat{\sigma}_1^{(N)} \end{pmatrix} \quad M_Y \begin{pmatrix} \sigma_8 \\ \sigma_9 \\ \sigma_{10} \end{pmatrix} = \begin{pmatrix} \hat{\sigma}_8^{(1)} \\ \hat{\sigma}_8^{(2)} \\ \ldots \\ \hat{\sigma}_8^{(N)} \end{pmatrix} \tag{2}$$

$$M_{XY} \begin{pmatrix} \sigma_3 \\ \sigma_4 \\ \sigma_6 \\ \sigma_7 \end{pmatrix} = \begin{pmatrix} \hat{\sigma}_3^{(1)} \\ \hat{\sigma}_3^{(2)} \\ \ldots \\ \hat{\sigma}_3^{(N)} \end{pmatrix} \tag{3}$$

where the matrices M_X, M_Y and M_{XY} are defined by:

$$M_X = \begin{pmatrix} R_{11}^{2(1)} & 2R_{11}^{(1)}R_{12}^{(1)} & R_{12}^{2(1)} \\ R_{11}^{2(2)} & 2R_{11}^{(2)}R_{12}^{(2)} & R_{12}^{2(2)} \\ \ldots & \ldots & \ldots \\ R_{11}^{2(N)} & 2R_{11}^{(N)}R_{12}^{(N)} & R_{12}^{2(N)} \end{pmatrix} \tag{4}$$

$$M_Y = \begin{pmatrix} R_{33}^{2(1)} & 2R_{33}^{(1)}R_{34}^{(1)} & R_{34}^{2(1)} \\ R_{33}^{2(2)} & 2R_{33}^{(2)}R_{34}^{(2)} & R_{34}^{2(2)} \\ \ldots & \ldots & \ldots \\ R_{33}^{2(N)} & 2R_{33}^{(N)}R_{34}^{(N)} & R_{34}^{2(N)} \end{pmatrix} \tag{5}$$

$$M_{XY} = \begin{pmatrix} R_{11}^{(1)}R_{33}^{(1)} & R_{11}^{(1)}R_{34}^{(1)} & R_{12}^{(1)}R_{33}^{(1)} & R_{12}^{(1)}R_{34}^{(1)} \\ R_{11}^{(2)}R_{33}^{(2)} & R_{11}^{(2)}R_{34}^{(2)} & R_{12}^{(2)}R_{33}^{(2)} & R_{12}^{(2)}R_{34}^{(2)} \\ \ldots & \ldots & \ldots & \ldots \\ R_{11}^{(N)}R_{33}^{(N)} & R_{11}^{(N)}R_{34}^{(N)} & R_{12}^{(N)}R_{33}^{(N)} & R_{12}^{(N)}R_{34}^{(N)} \end{pmatrix} \tag{6}$$

Analytical Conditions

Projected Emittance (2D) In the general case of N measurement stations, one has to consider the $N \times 3$ matrix M_X and its associated augmented $N \times 4$ matrix. The system (2) has a unique solution $(\sigma_1, \sigma_2, \sigma_5)$ if and only if the rank of both M_X and M_X^* matrices is equal to three. That means that the determinants of all 3×3 minors of M_X should not vanish, and the determinants of all 4×4 minors of M_X^* should be equal to zero. Define the 3×3 minors of the M_X matrix by the row indices (i, j, k), with $i, j, k = 1, 2 \ldots N$. In terms of the Twiss parameters the

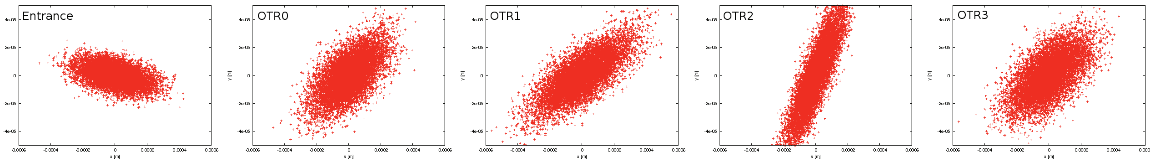

Figure 1: Beam transverse distribution at the entrance of the EXT line, coupled with $r = -0.4$, and at the four OTR stations.

determinants of such minors are written as

$$\Delta_{3x}(ijk) = 2\beta_x^{(i)}\beta_x^{(j)}\beta_x^{(k)} \sin\phi_x^{(ji)} \sin\phi_x^{(ki)} \sin\phi_x^{(kj)} \tag{7}$$

Analogously, the 4×4 minors of the augmented matrix M_X^* are characterized by the row indices (i, j, k, l). Its determinants are written as

$$\begin{aligned} A_{4x}(ijkl) = -\hat{\sigma}_1^{(i)}\Delta_{3x}(jkl) + \hat{\sigma}_1^{(j)}\Delta_{3x}(ikl) \\ -\hat{\sigma}_1^{(k)}\Delta_{3x}(ijl) + \hat{\sigma}_1^{(l)}\Delta_{3x}(ijk) \end{aligned} \tag{8}$$

The first condition is that $\Delta_3(ijk) \neq 0$, which is equivalent to the condition that the betatron phase advance differences should not be an integer multiple of π:

$$\phi_x^{(ji)} \neq n\pi \,, \forall(i,j) \tag{9}$$

This is the only required condition in the case of 3 measurement stations. For four or more stations, a second condition is required to get a unique solution:

$$\begin{aligned} -\hat{\sigma}_1^{(i)}\Delta_{3x}(jkl) + \hat{\sigma}_1^{(j)}\Delta_{3x}(ikl) \\ -\hat{\sigma}_1^{(k)}\Delta_{3x}(ijl) + \hat{\sigma}_1^{(l)}\Delta_{3x}(ijk) = 0 \,, \forall(i,j,k,l) \end{aligned} \tag{10}$$

One can see that Eq. (9) contains only optical restrictions about the betatron phase advances, which are easily understood: the measurement stations should be located at places where the phase advances correspond to different snapshots of the beam. If the number of stations is greater than 3, one should located them in such a way that this condition is fulfilled for any combination of three stations. Otherwise one deals with meaningless solutions, introducing unnecessary noise which eventually lead to unphysical results. Besides the phase advances, Eq. (10) involves also the amplitudes $\beta_x^{(i)}$ and the measured values of $\hat{\sigma}_1^{(i)}$. Obviously, due to the statistical variance of the latter, the equality implied by Eq. (10) cannot be exactly satisfied. One could calculate the variance of the lhs entering Eq. (10) by using standard error mechanism, and reject measured values leading to combinations greater than some previously fixed error bar. Finally, and for the same reasons mentioned for Eq. (9) to avoid unphysical results, the equality (10) should be satisfied for any combination of 4 stations.

An analogous condition for the vertical plane can be written by changing $\hat{\sigma}_1^{(i)}$ for $\hat{\sigma}_8^{(i)}$.

Coupling Terms Let us now consider the system (3). A minimum of four measurements are required to obtain the coupling terms $\sigma_3, \sigma_4, \sigma_6$ and σ_7. Besides the $N \times 4$ matrix M_{XY} one has to consider also its associated augmented $N \times 5$ matrix. The system (3) has a unique solution $(\sigma_3, \sigma_4, \sigma_6, \sigma_7)$ if and only if the rank of both M_X and M_X^* matrices is equal to four. That means that the determinants of all 4×4 minors of M_X should not vanish, and the determinants of all 5×5 minors of M_X^* should be equal to zero. Proceeding along the same lines as in the previous case for the projected emittances, we could write the determinants of the 4×4 minors of M_{XY} and the determinants of the 5×5 minors of the augmented matrix in terms of the Twiss parameters [4]. Therefore the necessary and sufficient conditions to have a unique solution are:

$$\begin{aligned} &\cos\left(\phi_x^{(ji)} + \phi_x^{(lk)}\right)\left[\cos\left(\phi_y^{(ki)} + \phi_y^{(lj)}\right) - \cos\left(\phi_y^{(ki)} - \phi_y^{(lj)}\right)\right] \\ &+\cos\left(\phi_x^{(ki)} + \phi_x^{(lj)}\right)\left[\cos\left(\phi_y^{(ji)} - \phi_y^{(lk)}\right) - \cos\left(\phi_y^{(ji)} + \phi_y^{(lk)}\right)\right] \\ &+\cos\left(\phi_x^{(ji)} - \phi_x^{(lk)}\right)\left[\cos\left(\phi_y^{(ji)} + \phi_y^{(lk)}\right) - \cos\left(\phi_y^{(ki)} + \phi_y^{(lj)}\right)\right] \\ &\neq 0 \,, \forall(i,j,k,l) \end{aligned} \tag{11}$$

This is the only condition required in the case of four measurement stations. Similarly to Eq. (9) , it restricts the optical properties. Notice that in the particular case where $\phi_x^{(ji)} = \phi_y^{(ji)}$ the system has no solution. When more than four measurement stations exist an additional condition is required:

$$\begin{aligned} -\hat{\sigma}_3^{(i)}\Delta_4(jklm) + \hat{\sigma}_3^{(j)}\Delta_4(iklm) - \hat{\sigma}_3^{(k)}\Delta_4(ijlm) \\ +\hat{\sigma}_3^{(l)}\Delta_4(ijkm) - \hat{\sigma}_3^{(m)}\Delta_4(ijkl) = 0 \,, \forall(i,j,k,l,m) \end{aligned} \tag{12}$$

respectively. Notice that the indices $i, j, \ldots$ entering these conditions represent each station. They take values from 1 to N and should be different from each other. As discussed for Eq. (9) and Eq. (10), these conditions should be satisfied for any combination of 4 or 5 station. Besides, the same argument concerning the experimental uncertainties of $\hat{\sigma}_3^{(i)}$ applies for Eq. (12).

TRACKING SIMULATIONS

We have performed realistic tracking simulations in high coupling scenarios in the EXT line of ATF2 to test the robustness of the method in the mi-OTR system located in this line. Simulations have been carried out using the ATF2 optics version V5.1 in MADX, and including multipoles. The coupling at the entrance of the EXT line has been generated assuming a Gaussian distribution in transversal directions. We can generate a 4D uncoupled Gaussian phase

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Table 1: Nominal optics parameters of the EXT line of ATF2, simulated beam sizes at the entrance of the EXT line and at each OTR station, and projected (2D) and intrinsic (4D) emittances.

Nominal optics V5.1							
	Entrance	OTR0	OTR1	OTR2	OTR3		
β_x (m)	6.849	6.305	10.654	4.125	7.458	ϵ_x (m rad)	2.00×10^{-9}
β_y (m)	2.936	6.190	4.343	10.971	5.430	ϵ_y (m rad)	1.18×10^{-11}
$\mu_x/2\pi$	0	2.884	2.905	3.014	3.104		
$\mu_y/2\pi$	0	2.146	2.200	2.277	2.402		
Tracking							
	$r = 0$					ϵ_x (m rad)	1.964×10^{-9}
σ_1 (m^2) $\times 10^{-8}$	1.375	1.291	2.177	0.827	1.480	ϵ_y (m rad)	1.196×10^{-11}
σ_8 (m^2) $\times 10^{-11}$	3.605	7.490	5.193	12.923	6.432	ϵ_1 (m rad)	1.964×10^{-9}
σ_3 (m^2) $\times 10^{-12}$	-5.002	2.588	-5.448	-8.517	5.076	ϵ_2 (m rad)	1.197×10^{-11}
	$r = -0.4$					ϵ_x (m rad)	1.963×10^{-9}
σ_1 (m^2) $\times 10^{-8}$	1.357	1.295	2.180	0.821	1.470	ϵ_y (m rad)	2.689×10^{-11}
σ_8 (m^2) $\times 10^{-11}$	4.292	19.068	20.141	64.508	20.968	ϵ_1 (m rad)	1.959×10^{-9}
σ_3 (m^2) $\times 10^{-9}$	-0.331	0.974	1.565	2.018	1.080	ϵ_2 (m rad)	1.240×10^{-11}
	$r = -0.8$					ϵ_x (m rad)	2.479×10^{-9}
σ_1 (m^2) $\times 10^{-8}$	1.346	1.289	2.168	0.821	2.168	ϵ_y (m rad)	NaN
σ_8 (m^2) $\times 10^{-11}$	10.351	46.230	62.413	234.41	62.413	ϵ_1 (m rad)	2.496×10^{-9}
σ_3 (m^2) $\times 10^{-9}$	-0.961	1.718	2.990	4.220	2.990	ϵ_2 (m rad)	NaN

space distribution (no dispersion) using the following expressions:

$$
\begin{aligned}
x &= [g_1]\sqrt{\beta_x\epsilon_x},\ x' = [g_2]\sqrt{\frac{\epsilon_x}{\beta_x}} - [g_1]\,\alpha_x\sqrt{\frac{\epsilon_x}{\beta_x}} \\
y &= [g_3]\sqrt{\beta_y\epsilon_y},\ y' = [g_4]\sqrt{\frac{\epsilon_y}{\beta_y}} - [g_3]\,\alpha_y\sqrt{\frac{\epsilon_y}{\beta_y}}
\end{aligned}
\tag{13}
$$

where $[g_1]$, $[g_2]$, $[g_3]$ and $[g_4]$ are standard normal random variables. A 4D coupled phase space can be obtained from the uncoupled one in a symplectic way by means of the following transformation [5]:

$$
\begin{pmatrix} x_c \\ x'_c \\ y_c \\ y'_c \end{pmatrix} = \begin{pmatrix} 1 & 0 & c_1 & c_2 \\ 0 & 1 & c_3 & c_4 \\ -c_4 & c_2 & 1 & 0 \\ c_3 & -c_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \tag{14}
$$

where four coupling parameters c_1, c_2, c_3 and c_4 are defined. Demanding this transformation matrix V to be symplectic, i.e. $\det V = 1$, we can write one parameter in terms of the other ones: $c_4 = c_2c_3/c_1$. In this scenario we can define [6] a coupling coefficient $r = \sigma_3/\sqrt{\sigma_1\sigma_8}$.

For instance, Fig. 1 shows the beam transverse distribution generated at the entrance of the EXT line with a coupling of $r = -0.4$ and the beam distribution from tracking simulations at the different OTR measurement stations. Table 1 summarizes the nominal optics parameters, the beam sizes from simulations and the 2D and 4D emittances calculated by solving the system of equations (2) and (3).

CONCLUSIONS

We have studied the mathematical conditions for the existence and unicity of solutions of the systems of equations involved in the process of emittance reconstruction. We have shown that there are four general conditions which should be satisfied to get physical solutions. The repetition of the measurements, i.e, statistics, only gives us a lower value of the variance of the measurement. The results of the present analysis will be very useful to better determine the location of emittance measurement stations in a design phase for the diagnostic sections of Future Linear Colliders. Simulations performed to test the robustness in high coupling scenarios agree with the analysis of the conditions.

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