ANALYTICAL ESTIMATIONS OF THE DYNAMIC APERTURES OF BEAMS WITH MOMENTUM DEVIATION AND APPLICATION IN FFAG*

Ming Xiao†, Jie Gao, IHEP, Beijing, China

Abstract

Analytical formulae for estimating the dynamic apertures of synchrotron particles has been well established. Based on the standard mapping, we extend the analytical formulae of dynamic aperture for off-momentum particles in circular accelerator. And we compare the analytical results with the simulation ones in the BEPC-II positron ring lattice under some conditions. What's more, we give the analytical formulae of dynamic aperture for FFAG in the similar way.

INTRODUCTION

Dynamic aperture, which is defined as the maximum phase-space amplitude when particles do not get lost as a consequence of single-particle-dynamics effects, is one of the major issues in accelerator design. The dynamic aperture would be very large if the machine consists of only nearly linear elements such as drift spaces and dipoles and quadrupoles. Though the nonlinear forces from all kinds of elements are very small, sometimes they are the murders of circulating particles in the pipe. It is badly important to estimate the dynamic aperture in a circular accelerator design.

However, the dynamic aperture is typically determined by numerical tracking till now. In [1], on the idea of comparing the single particle motion equations with standard mapping by virtue of the KAM theory and Chirikov criterion, it is the first time to estimate the dynamic aperture analytically under some assumptions.

HAMILTONIAN FORMALISM

It is convenient to use Hamiltonian method to deal with various nonlinear dynamical problems. In curvilinear coordinates, we give the Hamiltonian in a circular accelerator by using the arc length as the independent variable rather than time:

$$H_s = -(1+x/\rho)\left(P^2-(p_x-eA_x)^2-(p_y-eA_y)^2\right)^{1/2} - eA_s - e\Phi \quad (1)$$

where ρ is the radius of curvature and the torsion of the closed orbit is everywhere zero, P is the total mechanical momentum of the particle.

When the particle circulating in the accelerator ring is off-momentum, the Hamiltonian would have something different from the reference one. One could rewrite the Hamiltonian[2] including only one sextupole in the x plane

$$H = \frac{p_\beta^2}{2} - (1-\Delta)\left(K_x + \Delta SD\right)\frac{x_\beta^2}{2} + (1-\Delta)S\frac{x_\beta^3}{6} \quad (2)$$

where the quantity $\Delta \equiv (p-p_0)/p_0$ measures the deviation of the actual momentum from the momentum on the reference orbit, S is a periodic function and it is typically piecewise constant in the regions where the correction sextupoles are placed and zero elsewhere, $D(s)$ is the dispersion function in horizontal direction.

ANALYTICAL FORMULAE WITH DEPENDENT SEXTUPOLES

We assume that the contribution of nonlinear force from sextupoles and other kinds of multipoles in a circular accelerator can be made equivalent to a point sextupole and multipole. To facilitate the analytical treatment of the complicated problem, we assume that all the nonlinear forces just originate from sextupoles(no screw terms). The one dimensional Hamiltonian in action-angle variables for on-momentum particles is:

$$H = \frac{2\pi\nu}{L}J + \sum_{i,k}\frac{[2J\beta_x(s_i)]^{3/2}}{3}S_i\cos^3\Psi_i\delta(s-kL) \quad (3)$$

where L is the circumference of the ring, $\beta_x(s_i)$ and S_i represent the beta function and sextupole strength at the position s_i respectively. And $\Psi_i = \Psi + \Delta\Psi_i$, J and Ψ are the action-angle variables at the reference position, $\Delta\Psi_i$ is the phase advance of the sextupole at the position s_i away from the reference position. On the basis of the above Hamiltonian, the dynamic equations of single particle in a circular accelerator in the canonical action-angle variables are:

$$\frac{\mathrm{d}J}{\mathrm{d}s} = -\sum_{i,k}\frac{(2J)^{3/2}}{3}A_i\frac{\mathrm{d}}{\mathrm{d}\Psi}\cos^3\Psi_i\delta(s-kL) \quad (4)$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}s} = \frac{2\pi\nu}{L} + \sqrt{2J}\sum_{i,k}A_i\cos^3\Psi_i\delta(s-kL) \quad (5)$$

Here we use $A_i = \beta_x(s_i)^{3/2}S_i$ to make the equations compact and it will be very convenient for the following derivation.

Next, in order to analyse the possibilities of stochasticity, we change the differential equations eq.(4) and eq.(5) into difference ones[3][4], which have some distinguish with that in [1], because here we put all the nonlinear effects as a whole one. By using the Fourier analytical method to $\cos^3\Psi$ and combining with trigonometric relation, one

* The Natural Science Foundation of China under the Grant No.11175192

† xiaoming@ihep.ac.cn

could have different equation associated with $\overline{J}$ and $\overline{\Psi}$:

$$\overline{J} = J + \frac{(2J)^{3/2}}{4}\sqrt{B^2+C^2}\sin(3\Psi) \quad (6)$$

$$\overline{\Psi} = \Psi + \frac{1}{4}\sqrt{\frac{2}{J}}\sqrt{B^2+C^2}\overline{J} \quad (7)$$

where $B = \sum_i A_i \cos 3\Delta\Psi_i$ and $C = \sum_i A_i \sin 3\Delta\Psi_i$. Here we assume the cosine term $\cos 3\Psi$ keep constant to 1.

Till now, we have made the equations be the same pattern with the so-called standard mapping, which is expressed as

$$\overline{I} = I + K_0 \sin\theta \quad (8)$$

$$\overline{\theta} = \theta + \overline{I} \quad (9)$$

Comparing the eq.(6) and eq.(7) with standard mapping, one could easily reflect that $K_0 = \frac{3}{4}J(B^2+C^2)$. By virtue of the Chirikov criterion[3], it is well known that resonance overlapping occurs when $|K_0| \geq 0.97164$, which leads to particles' stochastic motions and diffusion processes[5]. So from $\frac{3}{4}J(B^2+C^2) \leq K_0 \sim 1$, one gets the maximum J corresponding to $m/3$ resonance:

$$J \leq J_{max} = \frac{4}{3(B^2+C^2)} \quad (10)$$

Then the dynamic aperture in the x plane is

$$A_{dyna,sext,x} = \sqrt{2J_{max}\beta_x(s)} = \sqrt{\frac{8\beta_x(s)}{3(B^2+C^2)}} \quad (11)$$

eq.(11) describes the total dynamic aperture determined just by sextupoles especially concerning with dependent influence.

DYNAMIC APERTURE WITH MOMENTUM DEVIATION

Now we would start our journey to search the analytical formulae of dynamic aperture for off-momentum particles. To start with we consider the horizontal motion of the reference particle in the horizontal plane within a series of sextupoles, so one gets the new Hamiltonian as:

$$H = \frac{p_\beta^2}{2} - (1-\Delta)\left(K_x + \Delta\sum_i SD\right)\frac{x_\beta^2}{2} + (1-\Delta)\sum_i S\frac{x_\beta^3}{6} \quad (12)$$

The Hamiltonian could be separated into two parts, i.e. the linear part associated with the former two terms and the nonlinear part associated with the last term. Compared with solution to $H = \frac{1}{2}p^2 + \frac{K(s)}{2}x^2$, one could get the solution to $H = \frac{p_\beta^2}{2} - (1-\Delta)(K_x + \sum_i S\Delta D)\frac{x_\beta^2}{2}$, which would have the form:

$$x_\beta = \sqrt{2J\tilde{\beta}_x(s)}\cos\left(\Psi - \frac{2\pi\nu}{L}s + \int_0^s \frac{ds'}{\tilde{\beta}_x(s')}\right) \quad (13)$$

where J and Ψ are the relevant action-angle variables. And the $\tilde{\beta}_x(s)$ is a new beta function connected with Δ, which could be calculated from

$$\tilde{\beta}_x(s) = \beta_x(s)\left(1 + \frac{1}{2\sin(2\pi\nu)}\oint \beta_x(t)k\cos\alpha dt\right) \quad (14)$$

where $k = \Delta K_x(t) - (1-\Delta)\Delta\sum_i D(t)S(t)$ is something like quadrupole field error, $\alpha = 2|\Psi(t) - \Psi(s)| - 2\pi\nu$. One may regard $\tilde{\beta}_x(s)$ as the equivalent beta function tasted by the off-momentum particles. Naturally, in order to calculate the integral more accurate in a convenient way, one could make the quadrupole and sextupole equivalent to delta function with finite effect length, e.g. it is reasonable to replace the $S(t)$ by $S(s_i)L_{s_i}\delta(s_i - t)$, where L_{s_i} is the effect length of the sextupole at the position s_i.

Taking the eq.(13) into eq.(12), the new Hamiltonian expressed in action-angle variables is:

$$H = \frac{2\pi\nu}{L}J + (1-\Delta)\sum_{i,k}\frac{[2J\tilde{\beta}_x(s_i)]^{3/2}}{3}S_i\cos^3\Psi_i\delta(s-kL) \quad (15)$$

The above Hamiltonian just has a difference beta function with that in eq.(3), we get the relationship between dynamic aperture of on-momentum particles and that of off-momentum ones easily:

$$A_{dyna,sext,\Delta} = \frac{1}{1-\Delta}\sqrt{\frac{8\tilde{\beta}_x(s)}{3(B^2+C^2)}} = \Omega \times A_{dyna,sext} \quad (16)$$

Here we call Ω the modulation factor. It is clear to tell that the dynamic aperture for off-momentum particles is modulated by both the momentum deviation and the linear lattice's characteristic.

COMPARED WITH SIMULATION RESULTS

Now we would apply eq.(11) and eq.(16) to a real machine. Here we choose the BEPC-II positron ring lattice as the object to compare the analytical results with simulation ones by using SAD code.

The BEPC-II, whose design energy is 1.89GeV, has a circumference of 237m long. And the positron ring lattice consists of 36 sextupoles in four families standing in symmetry positions.

Here the nonlinear forces are just from sextupoles. We set the tune be (6.5080, 5.5699) and choose the interaction point as the reference point. The results between the simulation results and our analytical estimation results are show in Figure 1.

It is clear to see that the analytical estimation result is quite good with simulation ones when the momentum deviation is quite small. Even though the difference between the two lines is a little unsatisfactory at very large momentum deviations, the analytical one indicates the right trends of dynamic aperture along with momentum deviation. From the picture, it is easy to see that the dynamic aperture may not be the largest for on-momentum particles, and the dynamic aperture would decrease when the momentum deviation is too large.

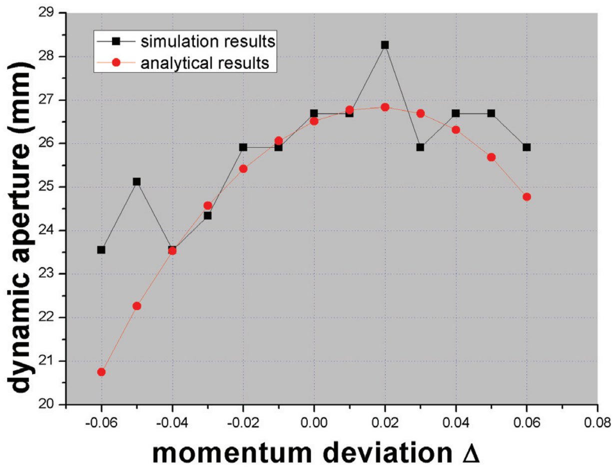

Figure 1: Results of horizontal dynamical aperture in both simulation method and analytical method at BEPC-II positron ring.

APPLICATION IN FFAG

FFAG is fixed field alternating gradient accelerator, and there are two types of FFAG, i.e. scaling FFAG and non-scaling FFAG. The scaling ones were proposed by Ohkawa, Kolomersky, Symon and Kerst in 1953[8] while the non-scaling ones were put forward by Mills and Johnstone in 1997[9] for dealing with the rapid acceleration ($\leq$ 20 turns) for muons.

For a linear non-scaling FFAG accelerator, the field is combined dipole with quadrupole. Usually, sextupoles are needed to correct the chromaticity, so eq.(11) would be suitable to estimate the dynamic aperture.

However, scaling FFAG accelerators consist of nonlinear fields, which is required as[10]

$$B(r) = B_0(r_0)\left(\frac{r}{r_0}\right)^k \quad (17)$$

where $B_0(r_0)$ is the reference field strength at r_0 and k is called the field index. Gradient is determined by focusing condition, not by isochronous condition. Here we consider a scaling radial FFAG as an example, which consists of N period of DFD structure. We assume that the F and D magnets have the same index number k with opposite magnet field direction. Expand the field as follows:

$$B(r) = B_0(r_0)\left[1+k\frac{\Delta r}{r_0}+\frac{k(k-1)}{2}\left(\frac{\Delta r}{r_0}\right)^2+\cdots\right] \quad (18)$$

We take the lowest nonlinear term into consideration and assume the nonlinear components act as delta functions. As the nonlinear effects are everywhere along the magnet in N families, the nonlinear force should have an influence with each other. So the Hamiltonian should be similar to eq.(3) just by replacing the sextupoles' strengths with new equivalent sextupoles' strengths:

$$H = \frac{2\pi\nu}{L}J+\sum_{i,k}\frac{[2J\beta_x(s_i)]^{3/2}}{3}\frac{k(k-1)}{2\rho r_0^2}\cos^3\Psi_i\delta(s-kL) \quad (19)$$

Repeat the process of getting dynamic equations in action-angle variables and comparing them with standard mapping, the dynamic aperture due to third-order effect is easily derived:

$$A_{dyna,third,x} = \frac{2\rho r_0^2}{k(k-1)}\sqrt{\frac{8\beta_x(s)}{3(\tilde{B}^2+\tilde{C}^2)}} \quad (20)$$

where in the new formulae $\tilde{B} = \sum_i \beta_x^{3/2}(s_i)\cos 3\Delta\Psi_i$ and $\tilde{C} = \sum_i \beta_x^{3/2}(s_i)\sin 3\Delta\Psi_i$. What's more, according to eq.(16), the modulation factor Ω will be useful to describe the dynamic aperture with momentum deviation:

$$A_{dyna,third,\Delta,x} = \Omega \times A_{dyna,third,x} \quad (21)$$

At the same time, one should take the high order effects into consideration to get more accurate dynamic aperture estimation for scaling FFAGs.

CONCLUSION

Based on the standard mapping, we extend a new formula for estimating dynamic aperture just caused by dependent sextupoles within momentum deviation. The result is quite good in BEPC-II positron ring. Also, we apply this method in scaling FFAG especially and give the analytical formula in third order.

ACKNOWLEDGEMENTS

The authors thank Yuan Zhang for his providing of the BEPC-II positron ring lattice and his help in using SAD. And we appreciate the discussions with Dou Wang and Yiwei Wang.

REFERENCES

[1] J. Gao, Nucl. Instr. and Methods, **A451** (2000) 545.
[2] R. Ruth, AIP conference proceedings, No. 153, p. 150.
[3] B.V. Chirikov, PR 52 (1979) 263.
[4] R.Z. Sagdeev, D.A. Usikov, and G.M. Zaslavsky, "Nonlinear Physics, from the pendulum to turbulence and chaos", Harwood Academic Publishers, 1988.
[5] J.M. Green, J. Math. Phys. 20 (1979), p. 1183.
[6] K. Oide and H. Koiso, Phys. Rev. E47, 2010 (1993).
[7] H. Wiedemann, "Particle Accelerator Physics II", Springer-Verlag, 2007.
[8] K.R. Symon et al, PR 103 (1956) 1837.
[9] C. Johnstone, Nucl. Instr. and Methods, **A503** (2003) 301.
[10] S. Machida, Nucl. Instr. and Methods, **A503** (2003) 322.