TWO AND THREE DIMENSIONAL MODELS FOR ANALYTICAL AND NUMERICAL SPACE CHARGE SIMULATION

S. Andrianov*, N.Kulabukhova †, SPbSU, SPb, Russia

Abstract

In this article there is described an analytical approach to describe the self-field of two- and three dimensional ellipsoidal presentation of space charge distribution. The corresponding results can be evaluated in both numerical and the analytic presentation for some model distributions of charge. The corresponding results can be embedded in the Lie formalism used to describe the map for the beam dynamics. The corresponding linear and nonlinear maps are evaluated in terms of the matrix representation of the evolution operator of the beam. Appropriate solutions for nonlinear differential equations are based on a prediction-correction method (the converging recursive procedure). These solutions are compared with the Vlasov equation solutions. A special software package for the described approach is presented.

INTRODUCTION

It is well known that the envelope equations for continuous beam with uniform charge density and elliptical cross-section were first derived by Kapchinsky and Vladimirsky (KV). This very useful result has been put into different approaches to charged beams description with any charge distribution with elliptical symmetry. More over this is also true in practice for three dimensional bunched beams with ellipsoidal symmetry. The utility of this rms approach was first demonstrated by Lapostolle for stationary distributions. Subsequently, Gluckstern [1] proved that rms version of KV-equations remain valid for all continuous beams with ellipsoidal form. Here we describe the approach based on these ideas for description of nonlinear space charge forces using ellipsoidal presentation of a space charge distribution. The purpose of analytical models is connected with necessity to improve the efficiency of numerical calculations (especially with the use of parallel and distributed computing systems), and on the other with providing a detailed analysis of the impact on the beam dynamics of various parameters (both the control system itself and the beam parameters). In this paper, we describe an approach to construct analytical expressions for the electric field produced by the beam particles. These expressions may be derived using the matrix formalism for a trajectory analysis [2, 3], and in terms of the envelope of the beam and/or the distribution function (in accordance with the Vlasov-Maxwell equations) [4].

* sandrianov@yandex.ru
† kulabukhova.nv@gmail.com

THE FORMS OF BEAM DESCRIPTION

In this paper, we develop the main conditions for the field, generated by the beam, as set out in [5], but taking into account the three-dimensional distribution of space charge in a bunched beam. It should be noted that a similar approach in the two-dimensional case allowed us to not only build a general analytical expression for a wide class of distributions of the beam, but also to integrate the expressions in the appropriate implementation of the perturbation theory, commonly used in beam physics [5]. In this paper, we focus on how the use of the matrix formalism [3] for Lie algebraic methods [6] in the event of calculating the self-field of the beam. This approach allows not only to carry out numerical experiments, but also to provide accurate analysis of the impact of different effects with the use of ready-made modules in accordance with the LEGO-objects (see, eg, [7]). As in [3] we use the method of calculation in symbolic form the components of the tension in two dimensions, for various models of the distribution of the transverse charge density $\rho(x, y)$, where x, y are transverse coordinates in according to the Ferrers'a integrals technology [8]. As a result, the expressions for the electric field of the beam we form the total field as the sum of external and the self-field of the beam, which can be written as

$$\mathbf{E}(x, y, s) == \sum_{k=1}^{\infty} \left(\mathbf{E}^k_{\text{out}}(x, y, s) + \mathbf{E}^k_{\text{self}}(x, y, s) \right), \quad (1)$$

where $\mathbf{E}^k$ contains the members of k-th order according the variables x, y, correspondingly. Similar presentation allows us to embed the total field in the general equation describing the dynamics of the particles in accordance with the matrix formalism

$$\frac{d\mathbf{X}}{ds} = \sum_{k=1}^{\infty} \left(\mathbb{P}^{[1k]}_{\text{out}}(s) + \mathbb{P}^{[1k]}_{\text{self}}(s) \right) \mathbf{X}^{[k]}, \quad (2)$$

where $\mathbf{X}^{[k]}$ the vector of k-th phase moments ($\dim \mathbf{X}^{[k]} = \binom{n+k-1}{k}$) [3].

1. Trajectory analysis. In this case the beam is presented as a particles assemble and can be written using the following matrix $\mathbb{X}^N = \{\mathbf{X}^1, \ldots, \mathbf{X}^N\}$, where $\mathbf{X}^k$ is a phase vector of k-th particle and N is a number of particles.

2. Beam envelope dynamics. In this case the beam is described in the terms of envelope matrices [3].

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3. Distribution function dynamics. In this case one present the beam in the terms of a distribution function, which satisfies to the Maxwell-Vlasov equations system.

THE SOLUTIONS OF MOTION EQUATIONS

The above mentioned types of beam presentations allow us to write the corresponding solutions in the universal form using the matrix formalism for Lie maps [3]. The solution of eq. 2 can be written in the form

$$\mathbf{X}(s)=\sum_{k=1}^{\infty}\mathbb{R}^{[1k]}(s|s_0)\mathbf{X}_0^{[k]}, \tag{3}$$

where $\mathbf{X}_0=\mathbf{X}(s=0)$ and matrices $\mathbb{R}^{[1k]}$ ($\dim\mathbb{R}^{[1k]}=2n\times\binom{n+k-1}{k}$) under $k\geq 2$ describe the nonlinear members in the our expansion of the desired solution of motion equation (2).

The solution (3) can be used for creation of all necessary objects for our approach. We should note that the procedure of beam evolution modeling (using the space charge forces) can be written in the following terms of beam descriptions. The first type of beam description leads us to

$$\mathbb{X}(s)=\sum_{k=1}^{\infty}\mathbb{R}^{[1k]}(s|s_0)\mathbb{X}_0^{[k]}. \tag{4}$$

The usage of envelope beam presentation allows us to write expanded envelope matrices $\mathfrak{S}^{ik}$, $i,k\leq 1$:

$$\mathfrak{S}^{ik}(s)=\sum_{l=i}^{\infty}\sum_{j=k}^{\infty}\mathbb{R}^{il}(s|s_0)\mathfrak{S}_0^{lj}\left(\mathbb{R}^{kj}(s|s_0)\right)^{\mathrm{T}}, \tag{5}$$

where $\mathfrak{S}^{ik}(s_0)$ the initial envelope matrices $\mathfrak{S}^{ik}(s_0)$ can be calculated according to

$$\mathfrak{S}^{ik}(s_0)=\int\limits_{\mathfrak{M}(s_0)}f(\mathbf{X},s_0)\mathbf{X}^{[i]}\left(\mathbf{X}^{[k]}\right)^{\mathrm{T}}d\mathbf{X}, \tag{6}$$

where $f(\mathbf{X},s_0)$ is a distribution function for our beam at the moment s_0, one can evaluate the matrices $\mathfrak{S}^{ik}$ up to necessary order of truncation in according to (5).

The third case leads us the following form of the desired solutions

$$\begin{aligned}
f(\mathbf{X},s) &= f_0\left(\mathcal{M}^{-1}\circ\mathbf{X}\right)=\sum_{k=0}^{\infty}\mathcal{M}^{-1}\circ\left(\mathbf{F}_k^0\right)^{\mathrm{T}}\cdot\mathbf{X}^{[k]}\\
&= \sum_{k=0}^{\infty}\left(\mathbf{F}_k^0\right)^{\mathrm{T}}\cdot\sum_{l=k}^{\infty}\mathbb{T}^{kl}\cdot\mathbf{X}^{[l]}=\sum_{k=0}^{\infty}\mathbf{F}_k^{\mathrm{T}}\cdot\mathbf{X}^{[k]},\\
\mathbf{F}_0 &= \mathbf{F}_0^0,\quad \mathbf{F}_k=\sum_{l=1}^{k}\left(\mathbb{T}^{kl}\right)^{\mathrm{T}}\mathbf{F}_l^0,\quad k\geq 1.
\end{aligned}$$

and the matrices $\mathbb{T}^{kl}$ can be evaluated from matrices $\mathbb{R}^{kl}$ using the well known generalized Gauss algorithm for inversion of block matrices.

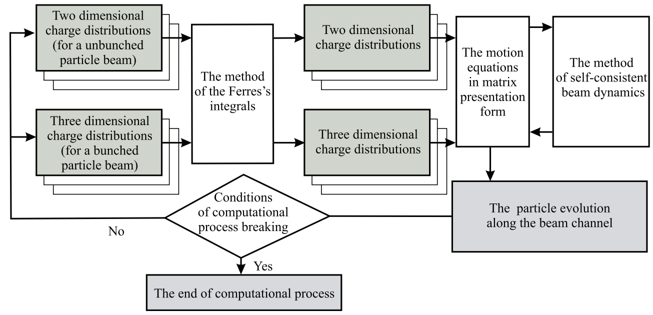

Figure 1: The scheme of construction of self-consistent beam particle evolution

THE SOLUTIONS OF MOTION EQUATIONS IN THE PRESENCE OF SPACE CHARGE FORCES

The above described forms of beam presentation usually are used for beam description both without influence of the self-field of the beam and in the presence of space charge forces. In this case we should use the special self-consistent method [3] and Fig.1. Let consider the basic features of this method on the example of the envelope matrices. Step 0. At first we should calculate $\mathfrak{S}_0^{ik}$, $i,k=\overline{1,N}$ according to the following formulae (here N is an order of series truncation):

$$\mathfrak{S}_0^{ik}=\int\limits_{\mathfrak{M}_0}f_0(\mathbf{X})\mathbf{X}^{[i]}\left(\mathbf{X}^{[k]}\right)^{\mathrm{T}}d\mathbf{X}.$$

As a matrix form $\mathbb{A}_0$ let choose $\left(\mathfrak{S}_0^{11}\right)^{-1}$ or the matrix $\mathfrak{S}_0^{-1}$, if the initial set $\mathfrak{M}_0$ is a ellipsoid with boundary $\mathbf{X}_0^{\mathrm{T}}\mathfrak{S}_0^{-1}\mathbf{X}_0=\varepsilon$. Then we constraint an approximating function $\varphi_0(\varkappa_0^2)\approx f_0(\mathbf{X}_0)$, $\varkappa_0^2=\mathbf{X}_0^{\mathrm{T}}\mathbb{A}_0\mathbf{X}_0$.

Step 1. We calculate the following block-matrices $\mathbb{P}^{1k}\left(\mathbf{B}^{\mathrm{ext}},\,\mathbf{E}^{\mathrm{ext}},s\right)$:

$$\mathbb{N}_1^{1k}=\mathbb{P}^{1k}\left(\mathbf{B}^{\mathrm{ext}},\mathbf{E}^{\mathrm{ext}},s\right).$$

Step 2. Compute the electrical field in the form $\mathbf{E}^{\mathrm{self}}=\mathbf{E}\left(\varphi_0(\varkappa_0^2)\right)$ according to some formulae (for example, see [3]) for appropriate model functions.

Step 3. Compute the block-matrices for motion equation $\mathbb{P}^{1k}(\mathbf{E}^{\mathrm{self}},s)$: $\mathbb{N}_2^{1k}=\mathbb{P}^{1k}\left(\mathbf{E}^{\mathrm{self}},s\right)$.

Step 4. Compute the block-matrices $\mathbb{R}^{ik}$ for the corresponding solution $i\leq k\leq N$,

$$\mathbb{R}_1^{ik}=\mathbb{R}^{ik}\left(s\,|\,s_0;\{\mathbb{N}_1^{1l}\}\right),\quad l=\overline{1,k},$$

$$\mathbb{R}_2^{ik}=\mathbb{R}^{ik}\left(s\,|\,s_0;\{\mathbb{N}_2^{1l}\}\right),\quad \mathbb{R}_0^{ik}=\mathbb{R}_1^{ik}+\mathbb{R}_2^{ik}.$$

Step 5.Compute the block-matrices for the corresponding the envelope matrices $\mathfrak{S}_0^{ik}$:

$$\mathfrak{S}_0^{ik}=\sum_{l=i}^{\infty}\sum_{j=k}^{\infty}\mathbb{R}_0^{il}\mathfrak{S}_0^{lj}\left(\mathbb{R}_0^{jk}\right)^{\mathrm{T}}$$

S t e p 6. We computer the block-matrices for *virtual changing* of beam parameters in the evolution process $\mathfrak{S}_1^{ik} = \alpha\mathfrak{S}_0^{ik} + (1-\alpha)\,\mathfrak{S}_0^{ik}, \quad 0 < \alpha < 1.$

S t e p 7. We check the conditions (or similar, see [3])

$$\|\mathfrak{S}_1^{ik} - \mathfrak{S}_0^{ik}\|_c < \varepsilon^{ik}. \tag{7}$$

With due fulfilment of inequalities (7) the process is finished. In another case we count $\mathfrak{S}_0^{ik} = \mathfrak{S}_1^{ik}$ and proceed to the the step 8.

S t e p 8.We find an approximating function (see [3]) $\varphi\left(\varkappa^2\right)$ for the function $f(\mathbf{X}, s)$:

$$\varphi\left(\varkappa^2\right) \approx f_0\left(\mathcal{M}_0^{-1} \circ \mathbf{X}_0\right) = f_0\left(\sum_{i=1}^{\infty} \mathbb{T}_0^{1i}\mathbf{X}_0^{[i]}\right).$$

We suppose $\varphi_0\left(\varkappa^2\right) = \varphi\left(\varkappa^2\right)$ and proceed to the Step 2.

AN APPROXIMATION OF THE SELF-CONSISTENT POTENTIAL

The above described algorithm should be controlled using some additional models for space charge description. As an example of similar additional model we consider the Poisson's equation for the self-consistent potential of the beam in two dimensional transverse space

$$\frac{\partial^2 \mathcal{V}_{\rm self}}{\partial x^2} + \frac{\partial^2 \mathcal{V}_{\rm self}}{\partial y^2} = -\int f(x, x', y, y', s)\, dx'\, dy', \tag{8}$$

where $f(x, x', y, y', s)$ is the Vlasov function for our beam. Let $\mathcal{H}(x, x', y, y') = m^2c^4 + m\frac{x'^2 + y'^2}{2} + q\mathcal{V}(x, y)$ be a Hamiltonian for our beam, $\mathcal{V} = \mathcal{V}_{\rm out} + \mathcal{V}_{\rm self}$, $\mathcal{V}_{\rm out}$ – is a potential of an external field and $\mathcal{V}_{\rm self}$ – is a potential of an self-consistent field generating by our beam. After integrating we can write

$$\rho(x, y) = \frac{2\pi q f_0}{m_0\gamma}\left(H_0 - m_0^2\gamma^2c^4 - q\mathcal{V}(x, y)\right), \tag{9}$$

and $\mathcal{V}(x, y)$ is a potential of the beam. Let introduce

$$\mathcal{U} = \alpha(H_0' - q\mathcal{V}), \quad \alpha = -\frac{2\pi q f_0}{\varepsilon_0 m_0 \gamma} \tag{10}$$

and the equation (9) can be rewritten in the form

$$\frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2} = k^2\mathcal{U}, \tag{11}$$

where $k^2 = -\alpha q = -2\pi q^2 f_0/\left(m_0\varepsilon_0\gamma\right) \leq 0$. In the elliptical coordinates (λ, μ we can write

$$\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\lambda}\left(\frac{\sqrt{g}}{g_{\lambda\lambda}}\frac{\partial}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(\frac{\sqrt{g}}{g_{\mu\mu}}\frac{\partial}{\partial\mu}\right)\right]\mathcal{U} = k^2\mathcal{U}, \tag{12}$$

Introducing new variables $R^2 = a^2 + \lambda$, $t^2 = -b^2 - \mu$ (here $b^2 + \lambda^2 = b^2 - a^2 + R^2$, $a^2 + \mu = a^2 - b^2 - t^2$, $\lambda - \mu = R^2 + t^2 + b^2 - a^2$, $-b^2 > \mu > -a^2$, $0 < t^2 < a^2 - b^2$) and applying the method $\mathcal{U}(R, t) = \mathcal{V}(R)\Phi(t)$ one can obtain two

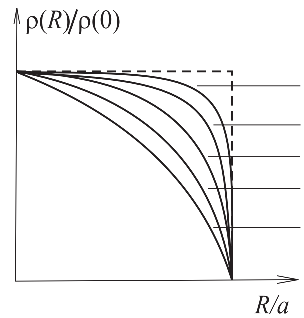

Figure 2: The density of the stationary charge distribution: *1* — $ka = 1$, *2* — $ka = 5$, *3* — $ka = 10$, *4* — $ka = 20$, *5* — $ka = 40$.

$$\frac{d^2\mathcal{V}}{dR^2} + \frac{R}{R^2 + b^2 - a^2}\frac{d\mathcal{V}}{dR} - \left(k^2 + \frac{\alpha}{R^2 + b^2 - a^2}\right)\mathcal{V} = 0, \tag{13}$$

$$\frac{d^2\Phi}{dt^2} + \frac{t}{t^2 + b^2 - a^2}\frac{d\Phi}{dt} - \left(k^2 - \frac{t^2k^2 - \alpha}{t^2 + b^2 - a^2}\right)\Phi = 0.$$

We should note that after the transition to polar coordinates ($t^2 \mapsto 0$) we obtain the well known Bessel equation of zeroth order

$$\frac{d^2\mathcal{V}}{dR^2} + \frac{1}{R}\frac{d\mathcal{V}}{dR} - k^2\,\mathcal{V} = 0. \tag{14}$$

Where a is a beam radius, and the second equation in (13) will be satisfied identically. The Fig.2 shows the profiles of stationary charge distribution (circular beam approximation) for the model "water bag" for different values of the parameter intensity $k\,a$.

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