ANALYSIS OF POSSIBLE FUNCTIONAL FORMS OF THE SCALING LAW FOR DYNAMIC APERTURE AS A FUNCTION OF TIME*

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Abstract

In recent studies, the evolution of the dynamic aperture with time has been fitted with a simple scaling law based on a limited number of free parameters. In this paper, different approaches to improve the numerical stability of the fit are presented, together with a new functional form.

INTRODUCTION

The time-dependence of the dynamic aperture (DA), i.e. the region of phase space that is stable over a fixed number of turns, has been proposed [1, 2] to satisfy

$$D(N) = D_\infty + \frac{b}{[\log(N)]^\kappa}, \tag{1}$$

where N represents the turn number and κ, D_∞, b are free parameters. Such a scaling is compatible with fundamental theorems in non-linear dynamics, such as the KAM [3] and Nekhoroshev [4, 5] theorems. Furthermore, this scaling has been used to model the time-evolution of beam losses in hadron machines without [6] and including beam-beam effects in the weak-strong regime [7], as well as the evolution of the luminosity in the LHC [8].

The data used are obtained from the SixTrack code [9], applying the standard protocol for tracking a LHC lattice representing the top energy configuration, including magnetic errors, and beam-beam effects in the weak-strong regime (see Ref. [7] for more details). Sixty realisations (called *seeds* below) of the magnetic errors have been considered, with a bunch charge of 0.02×10^{11} protons.

EVALUATION OF FIT PARAMETERS

Obtaining Values for κ

Difference Ratio Upon taking a difference ratio at turn times N_1, N_2, N_3 the parameters D_∞ and b are eliminated

$$\frac{D(N_1) - D(N_2)}{D(N_1) - D(N_3)} = \frac{1 - [\log(N_1)/\log(N_2)]^\kappa}{1 - [\log(N_1)/\log(N3]^\kappa}. \tag{2}$$

For fixed N_1 the LHS of Eq. (2) is then evaluated for all combinations $N_2, N_3 > N_1, N_2 \neq N_3$ and fitted to the RHS of Eq. (2). Varying N_1 yields a curve $\kappa = \kappa(N_1)$ and a corresponding curve for the residuals of the fits. Two different approaches are then used to obtain a value for κ.

1. A model $\kappa = const.$ is fitted through $\kappa = \kappa(N_1)$ including an error-weighting (labelled as method 3).
2. The value of $\kappa = \kappa(N_1)$ is chosen that gives the smallest residuals in the fit of the LHS of Eq. (2) (labelled as method 2).

Direct Fit with Varying κ Fixing the value of κ, one can evaluate (1) with only D_∞ and b as free parameters. By varying κ the residuals of the fits are obtained as a function of κ [6], and the value of κ is chosen that minimises the residuals.

Additional Methods for Evaluating D_∞ *and* b

Once a value for κ is determined, two more ways of obtaining D_∞ and b are investigated.

Obtaining b by Differences in DA Values Taking the difference in DA values at turn numbers N_1 and N_2 yields:

$$D(N_1) - D(N_2) = b\left\{[\log(N_1)]^{-\kappa} - [\log(N_2)]^{-\kappa}\right\} \tag{3}$$

A value for b can be computed in the following two ways:

1. Evaluate Eq. (3) for all $N_1 \neq N_2$ and compute the error-weighted average (labelled as method β).
2. Fit the RHS of Eq. (3) directly to the LHS with N_1 and N_2 as the independent variables and b as the fitting parameter (labelled as method γ).

Obtaining D_∞ by Ratio of DA Values Analogous to the procedure for b above, consider the ratio:

$$\frac{D(N_1)}{D(N_2)} = \frac{D_\infty/b - [\log(N_1)]^{-\kappa}}{D_\infty/b - [\log(N_2)]^{-\kappa}}, \tag{4}$$

from which the value of D_∞ can be derived as a function of b, N_1, N_2. Two approaches of determining D_∞ are:

1. Replace b in the expression of D_∞ from (4) by the corresponding solution of (3) to yield a value for $D_\infty(N_1, N_2)$, and compute the error-weighted average (labelled as method *b*).
2. Fit Eq. (1) to the data with κ and b fixed and D_∞ the fitting parameter (labelled as method *c*).

RESULTS

κ *Values from Different Methods*

Fig. 1 shows the distributions of the values of κ for each method. There is a general trend for κ to lie in an interval spanning a factor of at least two around $\kappa = -2$, with outliers. While all methods yield similar results, method 1 produces a systematically slightly different value of κ compared to the others.

* The HiLumi LHC Design Study is included in the High Luminosity LHC project and is partly funded by the European Commission within the Framework Programme 7 Capacities Specific Programme, Grant Agreement 284404.

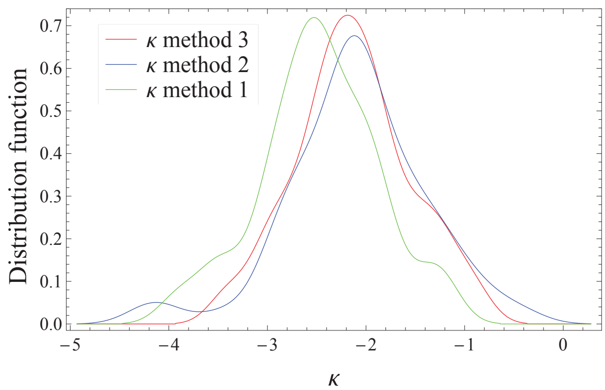

Figure 1: Distributions of κ values for different methods.

D_∞ Values from Different Methods

The results for D_∞ from the different methods are plotted in Fig. 2. Generally the different methods seem to give results of a similar order of magnitude, with typical variations being a few percent only. Nonetheless, the associated errors are increasing in size when moving from method a to method b and c. One can conclude that all the methods for determining κ and D_∞ generate compatible results. Moreover, D_∞ can be determined in a rather robust way independently of the details of the fit procedure.

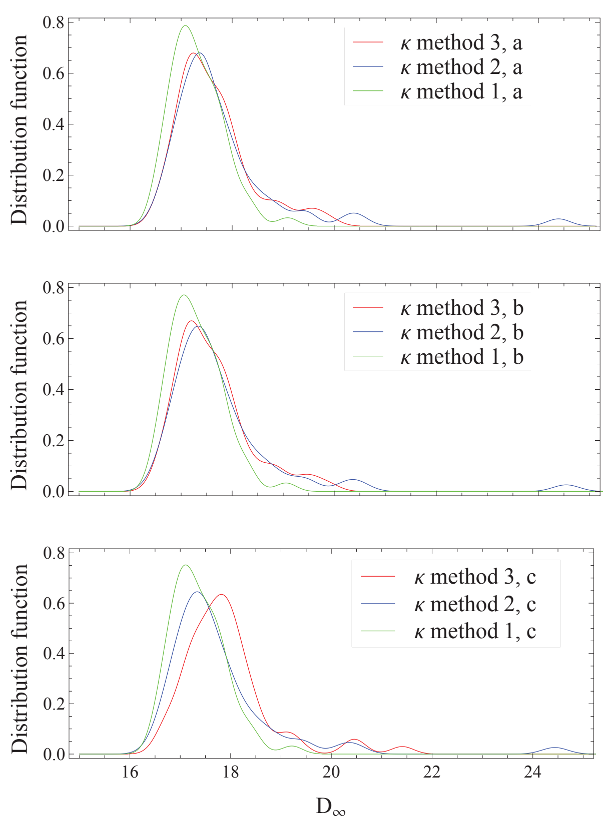

Figure 2: Distributions of D_∞ values and distributions from different methods.

b Values from Different Methods

The analysis of the distributions of b for the various methods (see Fig. 3) seems to indicate that the critical parameter is the way κ is determined. In fact, method 1 provides the narrower distribution even though all methods are affected by huge tails towards more negative values of b.

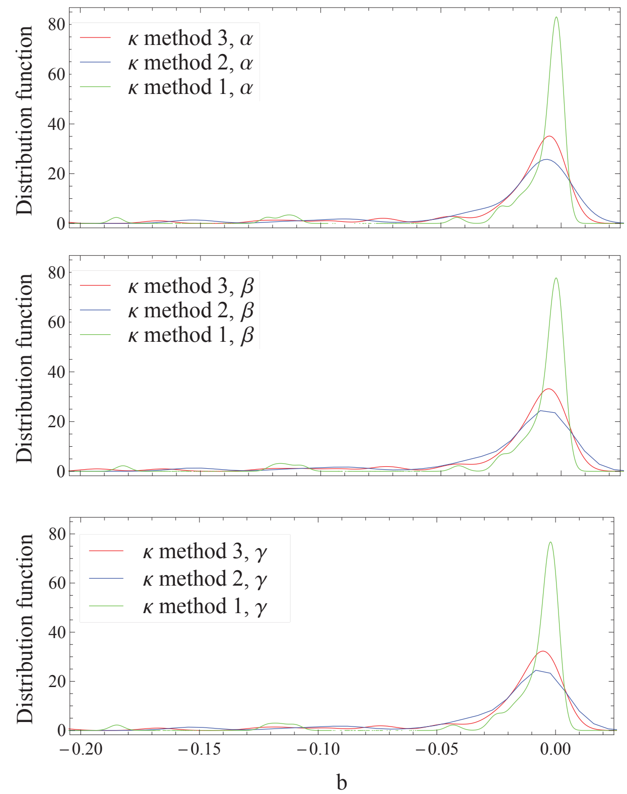

Figure 3: Distributions of b values from different methods.

Agreement Between Data and Fitting Methods

In Fig. 4 the DA values from the data are compared to the different fitting methods for a typical seed. The figure shows that the various methods yield comparable results for the time dependence of the DA values, which are compatible with the data.

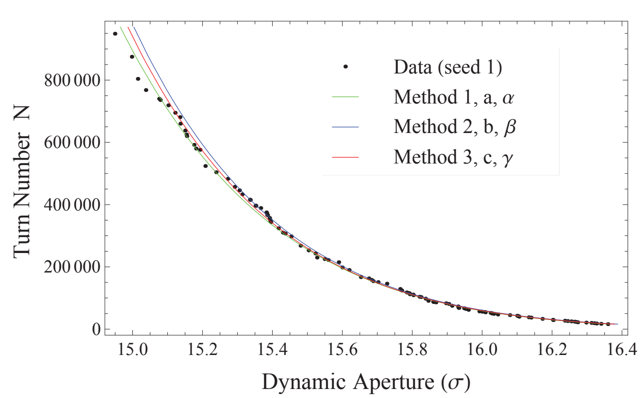

Figure 4: Numerical data and fit curves for seed 1.

ISBN 978-3-95450-122-9

ANALYSIS OF DIFFERENT MODELS

The analysis reported so far indicates that D_∞ is rather stable, while κ and b are varying much more, in particular b. Moreover, as both κ and b are part of the logarithmic dependence of DA a sort of compensation between them cannot be excluded, thus inducing large fluctuations over the seeds. Of course, there could be also the presence of a residual dependence of b, e.g., on κ.

In Fig. 5 the values of $\log|b|$ are plotted as a function of κ, showing a clear linear correlation. This feature is independent on the fit method.

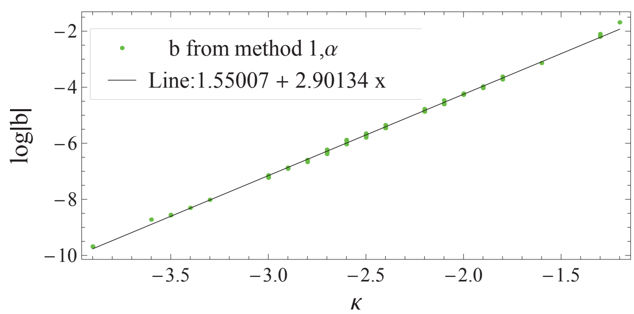

Figure 5: $\log|b|$ as a function of κ from method 1, α.

Therefore, one concludes that the relationship between b and κ is of the form:

$$\log|b| = \log|b_0| + b_1\kappa \qquad b = b_0 e^{b_1\kappa}. \tag{5}$$

The weighted average of the fit parameters b_0, b_1 from various methods is given by $\log|b_0| = 1.71 \pm 0.05$ and $b_1 = 2.97 \pm 0.02$. Therefore, the DA scaling model might be better described in the following form:

$$D(N) = D_\infty + \frac{b_0}{\left[\log\left(N^{e^{-b_1}}\right)\right]^\kappa}. \tag{6}$$

This new model has been applied to the same data set. Assuming $b_1 = 3$ one obtains the results shown in Fig. 6 for the parameters κ, D and b_0, respectively. The new model yields results for κ and D_∞ that are in good agreement with the old one, but that the value of b_0 is much more stable than the old fitting parameter b.

CONCLUSIONS

A detailed analysis of various approaches to fit the proposed scaling law (1) has been performed using a large number of numerical simulations. The original method used in Ref. [6] seems to perform better that the others. The analysis substantiates the validity of the dynamic aperture scaling law, but suggests a more suitable formulation:

$$D(N) = D_\infty + \frac{b_0}{\left[\log\left(N^{e^{-b_1}}\right)\right]^\kappa}. \tag{7}$$

For this new form the parameters D_∞ and κ show a stable behaviour in agreement with the initial formulation of

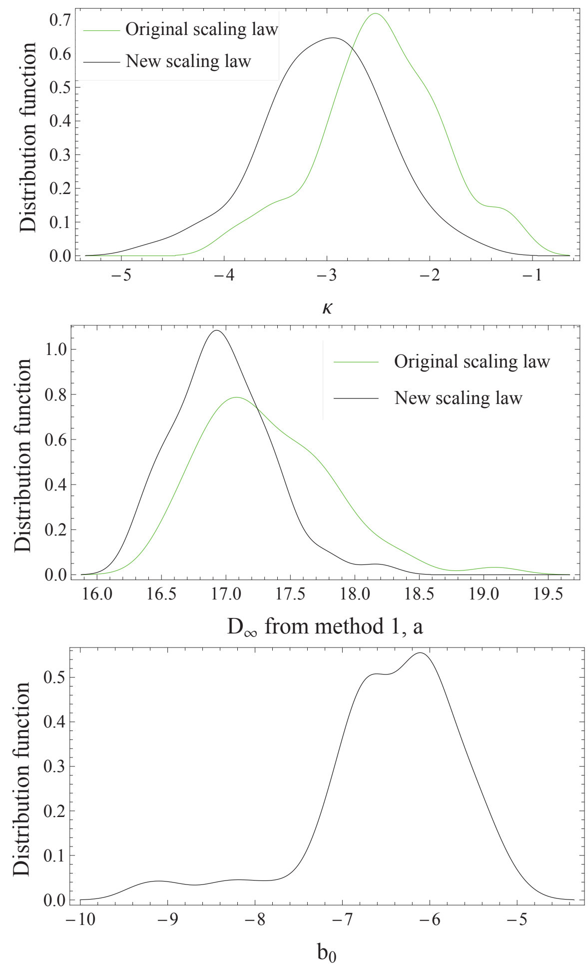

Figure 6: Distributions: κ, D_∞, b_0 for new and old models.

the DA scaling model, whereas the new parameter b_0 exhibits a much more stable behaviour than b in the form (1). In the applied fitting method the parameter b_1 was given the value 3, which was obtained from linear fits of $\log|b|$ as a function of κ. Further investigation is necessary in which this new revised form of the DA scaling law is applied to more data sets to determine whether the parameter b_1 can be assumed to be a constant.

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