QUANTITATIVE EVALUATION OF TRAPPING AND OVERALL EFFICIENCY FOR SIMPLE MODELS IN ONE-DEGREE OF FREEDOM

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Abstract

A key ingredient for the Multi-Turn Extraction (MTE) at the CERN Proton Synchrotron is the beam trapping in stable islands of transverse phase space. The control of the trapping process is essential for the quality of the final beam in terms of intensity sharing and emittance. In this paper, a method allowing an analytical estimation of the fraction of beam trapped into resonance islands as a function of the Hamiltonian parameters is presented for a very simple model of the dynamics (pendulum) and is extended to the case of the interpolating Hamiltonian of the Hénon model, the latter being a good 2D model of the MTE dynamics. The analytical results are compared with numerical simulations. Additional numerical simulations are presented for the minimum trapping amplitude and a fitted model is proposed. Results are discussed in detail.

MODELS

The first model considered in our study is governed by the pendulum-like Hamiltonian

$$H_1 = \frac{1}{2}\left[J - \delta(t)\right]^2 - \left[1 + \beta(t)\right]\cos\phi\,, \qquad (1)$$

in the case of adiabatic evolution of the parameters $\beta(t), \delta(t)$. A fixed point exists where $\dot{J} = \dot{\phi} = 0$, which holds when

$$J = \delta(t) \qquad \text{and} \qquad \phi = n\pi \qquad \text{with integer } n\,.$$

Fixed points with even n are stable, while if n is odd they are unstable. Connecting the unstable fixed point is the separatrix, with equation

$$J^*_\pm(t) = \delta(t) \pm 2\cos\left(\frac{\phi}{2}\right)\sqrt{1+\beta(t)} \qquad (2)$$

and energy $E^* = 1 + \beta(t)$. The surface area delimited by the separatrix (also called *resonance island*) and its time-derivative are given by

$$\Sigma_i = 16\sqrt{1+\beta(t)} \qquad \text{and} \qquad \dot{\Sigma}_i = \frac{8\,\dot{\beta}(t)}{\sqrt{1+\beta(t)}}\,. \quad (3)$$

The small-amplitude frequency of oscillation is given by

$$\omega(t) = \sqrt{1+\beta(t)}.$$

The second model under consideration is a quadratic polynomial 2D map, the Hénon map [1], which is customarily used as a simple, though realistic, model of the horizontal non-linear betatronic motion in a circular particle accelerator [2] and that has been studied recently in a similar context [3, 4] showing that the results of neo-adiabatic theory can be generalised to almost integrable 2D symplectic maps by using the interpolating Hamiltonian computed by Normal Forms Theory. The corresponding interpolating Hamiltonian [2] is given in action-angle variables by

$$H_2 = \epsilon\,J + \frac{\Omega_2}{2}\,J^2 + \epsilon\,|u_{0,3}|\,J^2\,\cos(4\phi) \qquad (4)$$

in which

$$\Omega_2 = -\frac{1}{16}\left(3\cot\frac{\omega}{2} + \cot\frac{3\omega}{2}\right) - \frac{3}{8}\,\kappa \qquad (5)$$

$$u_{0,3} = \frac{1}{16}\left(\cot\frac{\omega}{2} - \cot\frac{3\omega}{2}\right) - \frac{1}{8}\,\kappa \qquad (6)$$

$$\omega(t) = \epsilon(t) + \frac{\pi}{2} \qquad \text{with} \qquad \epsilon(t) = \frac{\Delta\omega}{T_{\text{max}}}\,t\,; \quad (7)$$

κ is a time-independent parameter in our analysis, while $\epsilon(t)$ allows sweeping through a non-linear resonance.

The factor multiplying ϕ is linked to the fourth-order resonance that is used in the standard application at CERN in the PS machine [5]. The fixed points are given by

$$J_\pm = -\frac{\epsilon}{\Omega_2 \pm 2\,|u_{0,3}|\,\epsilon} \qquad \phi_+ = k\,\frac{\pi}{2}, \quad \phi_- = \frac{\pi}{4} + k\,\frac{\pi}{2}$$

J_+ being stable and J_- unstable, while the separatrix reads

$$J^*_\pm = \frac{-\epsilon \pm 2\sqrt{\frac{|u_{0,3}|\,\epsilon^3\,\cos^2(2\phi)}{2\,|u_{0,3}|\,\epsilon - \Omega_2}}}{\Omega_2 + 2\,|u_{0,3}|\,\epsilon\,\cos(4\phi)}\,.$$

The expression of the island's surface ares is much more involved than in the case of the simple pendulum, nevertheless it can be written in closed form [6].

It is worth emphasising that while in the pendulum-like model the island size and position are independently controlled, in the Hénon map it is not possible to decouple the change in island size from the variation of island position.

TRAPPING PROBABILITY

The phase space of the considered models is divided by the separatrix into into three regions: upper (u), lower (ℓ), and resonance (i) that will be indexed by 1, 2 and 3, respectively. Assuming that a trajectory is initially in region 1, the trapping phenomenon occurs when we varies the parameters so to increase the areas of regions 2 and 3. The

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adiabatic theory estimates the trapping probabilities in regions (i) and (ℓ) [7]

$$P_i = \frac{\Theta_3}{\Theta_1} \quad \text{and} \quad P_\ell = \frac{\Theta_2}{\Theta_1} \quad \text{with} \tag{8}$$

$$\Theta_3 = \dot{\Sigma}_i > 0 \quad \Theta_2 = \dot{\Sigma}_\ell > 0 \quad \Theta_1 = \dot{\Sigma}_i + \dot{\Sigma}_\ell, \tag{9}$$

given the areas change *adiabatically*. Following the theory developed in, e.g., Ref. [8], it is possible to show that the trapping efficiency is indeed given by [6]

$$\frac{\Theta_3}{\Theta_1}(1 - 2k\sqrt{\varepsilon}) \tag{10}$$

for a suitable positive constant k, where ε is the time-derivative of the adiabatic parameter. According to the definitions used in this paper $\varepsilon = \dot{\epsilon} \simeq T_{\text{max}}^{-1}$. In this expression the impact of adiabaticity is explicit. The first term of Eq. (10) is equivalent to Eq. (8), while the second one is a correction term due to the finite variation velocity of the control parameter. Clearly, a reduced adiabaticity implies a reduced efficiency of the trapping process. Furthermore, by integrating over an ensemble of particles it is possible to estimate the fraction of the population that will be trapped as well as the scaling law of such a fraction as a function of the adiabaticity of the system. Another interesting feature is the estimate of the radius of the area around the origin of phase space where the adiabatic condition breaks down, which has a particular relevance in several practical applications. It can be shown [6] that such a radius scales as

$$R_{\text{min}} \simeq \varepsilon^{1/q} \tag{11}$$

where q is the order of the resonance under consideration.

TIME-DEPENDENT PENDULUM: NUMERICAL RESULTS

Several numerical simulations have been performed and the agreement between the formula (8) and the numerical results has been found to be very good [6].

Here, we report on a special case. We can compute the relationship between $\delta(t)$ and $\beta(t)$ necessary to keep $P_\ell = 0$ throughout the entire time-evolution of the system, so that the formulae (8) apply and predict that all particles interacting with the separatrix will be captured. This requires

$$\dot{\delta} = \frac{\dot{\Sigma}_i}{4\pi} = \frac{2\,\dot{\beta}(t)}{\pi\sqrt{1+\beta(t)}} \to \delta(t) = \delta_0 + \frac{4}{\pi}\left[\sqrt{1+\beta(t)}\right],$$

provided $\beta(0) = -1$.

Assume an initial distribution of particles given by

$$\rho(\phi, J) = \begin{cases} N/2\pi & \text{for } (\phi, J) \in [-\pi, \pi] \times [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

that δ increases linearly from $\delta(0) = 0.5$ to $\delta(T) = 1.5$, and that β varies quadratically from -1 to some $\beta(T)$ as

$$\delta(t) = \frac{1}{2} + \frac{t}{T} \qquad \beta(t) = \frac{\beta(T)+1}{T^2}\,t^2 - 1\,. \tag{12}$$

Then, after some algebraic manipulations it is possible to derive the trapping fraction τ corresponding to the parameters listed before and this gives

$$\tau = \begin{cases} \frac{4\sqrt{1+\beta(T)}}{\pi + 4\sqrt{1+\beta(T)}} & -1 \le \beta(T) \le \frac{\pi^2}{16} - 1\,, \\ \frac{4}{\pi}\sqrt{1+\beta(T)} - \frac{1}{2} & \frac{\pi^2}{16} - 1 \le \beta(T) \le \frac{9\pi^2}{64} - 1\,, \\ 1 & \beta(T) \ge \frac{9\pi^2}{64} - 1\,. \end{cases}$$

We set up simulations with $\delta(t)$ increasing linearly and $\beta(t)$ quadratic, using $T = 3500$ iterations. The fraction of total particles trapped by the island is displayed in Fig. 1 along with the predicted behaviour, which is in excellent agreement with the analytical predictions.

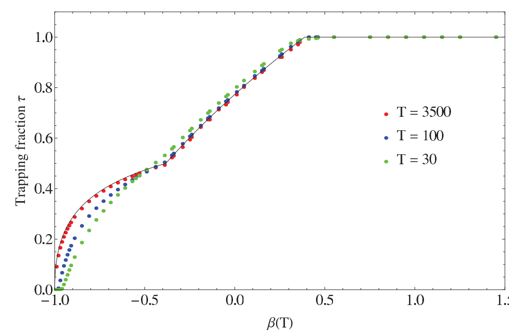

Figure 1: The results of the trapping fraction study when $\delta(t)$ increases linearly and $\beta(t)$ quadratically shows a close match with our predictions (solid line) when the adiabatic conditions is better fulfilled, i.e., when $T \to \infty$.

For shorter T (see Fig. 1) the motion of the separatrix is not adiabatic and the trapping is shown to be less efficient in the part where β varies quadratically. However, when β varies linearly the trapping seems even more efficient than the theoretical prediction. This might be due to the impact of the non-adiabaticity of the initial part of the trapping process that could have generated higher-density regions in the distribution, e.g., in the tails. This, in turn, could lead to an apparent increased trapping as in the theoretical model the particle distribution is always uniform.

Numerical tests have been performed to assess how efficient is the transport in the phase space for particles trapped in the islands. We define the quantity

$$\tau_{\text{travel}} = \frac{\#\text{ particles in island after travelling}}{\#\text{ particles initially trapped in island}}\,.$$

The theory [6] predicts a simple power law should exist between the travelling efficiency and the adiabatic parameter $\dot{\delta}/\omega^2$. To test this we set up initial conditions for 10^5 particles in the rectangle $[-\pi, \pi] \times [0, 1]$, and let the stable island begin as a zero-size slit at $J = 1/2$. Then we let the island grow adiabatically until achieving a given area (and thus capturing a given number of particles proportional to this area), and we subsequently let the island travel at various speeds by letting δ evolve from $1/2$ to 3.

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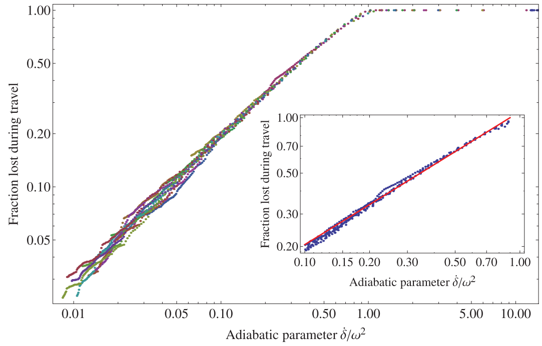

Figure 2: Travelling efficiency results after an island, full of particles, travels at various speeds. The theoretical prediction of a power-law dependence on the adiabatic parameter is confirmed. A zoom is shown in the inset.

At the end of each run, τ_{travel} has been computed. The results, for many different final island sizes and various travelling speeds (represented by the adiabatic parameter in the plot) are displayed in Fig. 2 and, fitting the regions between 10% and 90% efficiency, the following power law is found

$$1 - \tau_{\text{travel}} = (1.079 \pm 0.003)\, (\dot{\delta}/\omega^2)^{0.722 \pm 0.003} .$$

TIME-DEPENDENT HÉNON MAP: NUMERICAL RESULTS

The time-dependent Hénon map has been used to test the scaling laws (10) and (11). Massive numerical simulations have been performed in the past [3, 4] to assess the dependence of the trapping fraction τ as a function of the parameter κ and of the speed of resonance crossing which is chosen proportional to $1/T$ as in the other model. Both uniform and Gaussian distributions have been used and different values of the standard deviation σ have been selected. In Fig. 3 a summary of the numerical results is shown. The fit function is based in the expression (10) integrated

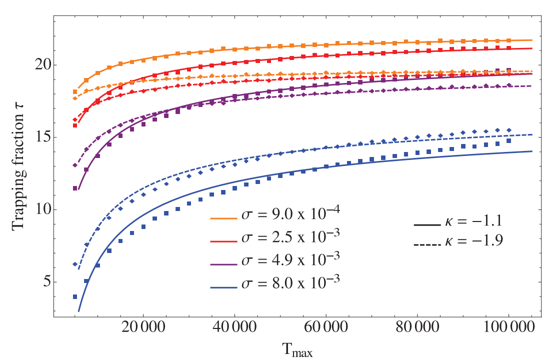

Figure 3: Trapping fraction τ as a function of the maximum number of iterations for different distributions and σ values. The fit curves are in very good agreement with the numerical data.

over the initial distribution, which suggests a dependence on $\simeq T_{\max}^{-1/2}$ and is nicely confirmed by the agreement between the fit and the numerical data. The dependence of τ on σ can also be predicted [6].

The other quantity of interest for our study is the scaling law of $R_{\min}$. It is clear that $R_{\min}$ is essential for any application aiming at a well-defined sharing of particles between islands and core. In Fig. 4 the numerical results are shown together with fit functions based on the scaling (11) for several values of the parameter κ and also resonance order q.

The log-log plot shows an excellent agreement between the scaling law and the numerical results.

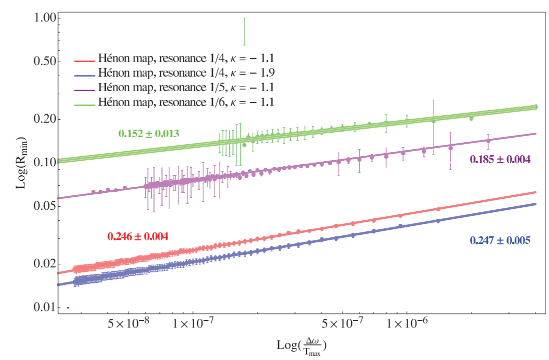

Figure 4: $R_{\min}$ as a function of the maximum iteration number in log-log scale. Different resonances and values of the parameter κ are shown together with the fit parameters.

CONCLUSION

We have investigated particle trapping into growing, travelling phase-space islands using two analytically approachable Hamiltonians. We determined how to predict accurately the fraction of particles which should be trapped into stable phase-space islands when the Hamiltonian changes adiabatically during the time-evolution, and we have compared the theoretical predictions of trapping behaviour in the near-adiabatic regime to numerous simulations showing an excellent agreement with the predicted scaling laws.

The next step will be to extend these results to Hamiltonian systems with two degrees of freedom in order to provide clear guidelines for the optimisation of the splitting process and hence propose a protocol for implementation in a real accelerator such as the CERN PS.

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