

A NEW METHOD TO INTEGRATE S-DEPENDENT HAMILTONIAN*

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Abstract

The present theory to obtain higher order terms of beam dynamics is mostly through Taylor expansion and differentiation, for example, the Lie transformation. When 3-dimensional Hamiltonian is being considered the operation of integration becomes necessary. In this paper we present a new integration theory, which leads to transfer maps for common accelerator elements based on 3-d Hamiltonians. Some immediate physics insight can be gained from this theory, for example, the kick-map theory which is used for insertion device design and modeling, is a first-order approximation in our approach.

INTRODUCTION

The basic idea of Lie Series is as follows. For any phase space coordinate z in a time-independent Hamiltonian system, where z stands for p or q , the canonical momentum and the coordinate, the differentiation can be written as $\frac{d}{dt}z = \frac{dp}{dt}\frac{\partial}{\partial p}z + \frac{dq}{dt}\frac{\partial}{\partial q}z = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p}z + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}z$ where the Hamilton's equations have been applied. If we use the Dragt notation $\frac{d}{dt}z = - : H : z$, then the solution can be expressed as $z(t) = e^{-t:H}z(t=0)$.

Unfortunately the number of terms increases exponentially with order of t in the expansion of $e^{-t:H}$; therefore for any practical calculations one has to truncate at a certain order. Several truncation methods have been proposed to preserve the symplectic Hamiltonian flow. One approach is to factorize the Hamiltonian into order by order generators [1]: $e^{H:L} = e^{H_2:L}e^{H_3:L}e^{H_4:L} \dots$, where H is the original Hamiltonian and H_i is an i th order (only) polynomial. The above factorization guarantees symplecticity up to a particular order in z , provided the Hamiltonian has no first-order terms. The new problem is that $e^{H_i:L}$ is an infinite series, except for $e^{H_2:L}$, of which one can always find a closed form. To truncate $e^{H_i:L}$ a refactorization approach is proposed to convert the higher order generators into integrable polynomials, for example $e^{H_3:L}e^{H_4:L} = e^{h_1:L}e^{h_2:L} \dots e^{h_7:L}$, where h_i are integrable polynomials and the coefficients are determined by H_3 and H_4 [2]. As pointed out in the same reference, the discarded terms are $\mathcal{O}(Lz^n)$ (i.e., higher order terms of Lz^n) for a n th order generator.

Another method is the so-called symplectic integrator [3, 4]. If the Hamiltonian can be written as $H = T(p) + V(q)$, then $e^{H:L}$ can be factorized as $\prod_{i=1}^k e^{c_i:T(p):L}e^{d_i:V(q):L}$, where c_i , d_i and k are coefficients depending upon the integration order n . At each step $e^{c_i:T(p):L}$ and $e^{d_i:V(q):L}$ can

be integrated exactly as a drift or a kick, respectively. With properly chosen coefficients the discarded terms are higher-order terms of L^n . However, because the Hamiltonian does not have constant terms the phase space terms will be on the same or higher order as L , hence the ignored terms are $\mathcal{O}(L^n z^n)$. This simple integration method is adopted by many codes; the disadvantage is that because the integration step must be small and the integration operation must be carried out for each particle, the tracking speed is relatively slow.

The above methods are developed for the s -independent Hamiltonian, and most of the results cannot be directly applied to a 3-d Hamiltonian. Much effort has been dedicated to the s -dependent problems; for instance, the recent work on the modeling of insertion devices [5, 6, 7]. These methods are either leading-order or second-order approximations. In this paper we present an exact solution to integrate through a 3-d Hamiltonian system. The method tracks particles like the truncated power series method [8] or the transfer matrix method [9]; however, it is extended to include all the effective higher-order terms. The coefficients need to be evaluated only once, and can be used for particles with different initial conditions. Therefore the computation speed is faster than the symplectic integrators. The method proposed in this paper leads to integration through the whole element (or period); therefore we believe it is different from the approach of COSY-INFINITY [8, 10].

THE INTEGRATION METHOD

First we define the differentiation in an s -dependent Hamiltonian system as

$$\frac{df}{ds} = [f, H] + \frac{\partial f}{\partial s} = - : H : f + \frac{\partial f}{\partial s} \equiv : H : f, \quad (1)$$

where f , g and H are functions of q , p , s , and $:H:$ = $- : H : + \frac{\partial}{\partial s}$ is defined as an operator.

Next we Taylor expand the phase space component $z_i(s_a + \Delta s)$ at s_a

$$\begin{aligned} & z_i(s_a + \Delta s) \\ &= z_i(s_a) + \frac{d}{ds}z_i(s)|_{s=s_a}\Delta s + \\ & \quad + \frac{1}{2}\frac{d^2}{ds^2}z_i(s)|_{s=s_a}\Delta s^2 + \dots \\ &= z_i(s_a) + \Delta s:H:z_i(s)|_{s=s_a} + \\ & \quad + \frac{1}{2}(\Delta s:H:)^2z_i(s)|_{s=s_a} + \dots, \\ &= \exp(\Delta s:H:)z_i(s)|_{s=s_a} \end{aligned} \quad (2)$$

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Therefore the transfer of phase space vector z from s_a to s_b can be written as

$$z(s_a + \Delta s) = \exp(\Delta s : \dot{H} :) z(s) |_{s=s_a}. \quad (3)$$

From now on we simplify the notation as $\exp(\Delta s : \dot{H} :) z(s) |_{s=s_a} = \exp(\Delta s : \dot{H} :) z(s_a)$.

To illustrate the integration method, we take the coordinate x as an example. From Eq. (2):

$$\begin{aligned} x(s_a + \Delta s) &= x(s_a) + \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s \\ &+ \frac{1}{2!} \left(\frac{\partial H}{\partial p_x} \frac{\partial}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial}{\partial p_x} + \frac{\partial}{\partial s} \right) \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s^2 \\ &+ \frac{1}{3!} \left(\frac{\partial H}{\partial p_x} \frac{\partial}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial}{\partial p_x} + \frac{\partial}{\partial s} \right) \\ &\left(\frac{\partial H}{\partial p_x} \frac{\partial}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial}{\partial p_x} + \frac{\partial}{\partial s} \right) \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s^3 + \dots \end{aligned} \quad (4)$$

Collecting all the $\frac{\partial^n}{\partial s^n} \frac{\partial H}{\partial p_x}$ terms,

$$\begin{aligned} &\tilde{T}_1 x \\ &+ \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s + \frac{1}{2!} \frac{\partial}{\partial s} \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s^2 \\ &+ \frac{1}{3!} \frac{\partial^2}{\partial s^2} \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s^3 + \dots \\ &= \frac{\partial}{\partial s} \int ds \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s + \frac{1}{2!} \frac{\partial^2}{\partial s^2} \int ds \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s^2 \\ &+ \frac{1}{3!} \frac{\partial^3}{\partial s^3} \int ds \frac{\partial H}{\partial p_x} |_{s=s_a} \Delta s^3 + \dots \end{aligned} \quad (5)$$

If we define $\frac{\partial}{\partial s} t_{x,1}(s) = \frac{\partial H}{\partial p_x}$, then Eq. (5) is the Taylor expansion of $t_{x,1}(s_a + \Delta s)$ at s_a , except for the first term, or

$$\tilde{T}_1 x = t_{x,1}(s_a + \Delta s) - t_{x,1}(s_a) = \int_{s_a}^{s_b} ds \frac{\partial H}{\partial p_x}. \quad (6)$$

The higher orders can be found iteratively. As a matter of fact, if we define

$$\begin{aligned} \ddot{I}_0 &= 1 \\ \ddot{I}_1 &= : \dot{H} : \\ \ddot{I}_n &= - : H : : \dot{H} : \quad \text{for } n > 1 \end{aligned} \quad (7)$$

and

$$\begin{aligned} &\tilde{T}_i(s_b, q(s_a), p(s_a), s_a) \\ &= \int_{s_a}^{s_b} ds_1 \int_{s_a}^{s_1} ds_2 \dots \int_{s_a}^{s_{i-1}} ds_i \ddot{I}_i, \end{aligned} \quad (8)$$

then

$$x(s_b) = \sum_{i=0}^{\infty} \tilde{T}_i x(s_a). \quad (9)$$

We note that in Eq. (9) the phase space coordinates p_i and q_i are evaluated at s_a , which is a result of the Taylor expansion.

Comparison with Eq. (2) leads to

$$\exp(\Delta s : \dot{H} :) = \sum_{i=0}^{\infty} \tilde{T}_i. \quad (10)$$

The validity of Eq. (9) can be verified by showing that

$$\begin{aligned} \frac{d}{ds_b} x(s_b) &= \sum_{i=0}^{\infty} \tilde{T}_i : \dot{H} : x(s_a) = \exp(\Delta s : \dot{H} :) : \dot{H} : x(s_a) \\ &= : \dot{H} : \exp(\Delta s : \dot{H} :) x(s_a) = : \dot{H} : x(s_b), \end{aligned} \quad (11)$$

which is consistent with our definitions.

Transformation of any phase space function $f(q(s), p(s), s)$ can be obtained in the same way, i.e.,

$$f(q(s_b), p(s_b), s_b) = \sum_{i=0}^{\infty} \tilde{T}_i f(q(s_a), p(s_a), s_a). \quad (12)$$

Here \tilde{T}_i and \ddot{I}_i are defined as transformation integration and differentiation operators.

We would like to point out that Equation (9) in integral form is the exact transformation for the s -dependent system. The advantage of (9) over (4) is that the s dependence of $\ddot{I}_i x$ is fully accounted by the integration, and the integration can be carried out for the entire element due to the inclusion of all the $\frac{\partial^n}{\partial s^n}$ terms; therefore (9) simplifies calculation and improves convergence.

COMPARISON WITH THE KICK-MAP THEORY

The kick-map theory [5] has been widely used for insertion-device design and modeling. In this section we are going to show that it is a first order (in L) approximation of the general solution presented in the previous section. We use the following vector potential

$$A_x = - \int h_s B_y ds \quad (13)$$

$$A_s = 0 \quad (14)$$

$$A_y = \int h_s B_x ds + g_s(x, y), \quad (15)$$

where $g_s(x, y) = - \int B_s(s=0) dx$. The s -independent B_s term is likely to be zero in an insertion device; therefore $g_s(x, y) \approx 0$.

The 2nd order Hamiltonian is given by

$$\begin{aligned} H_2 &\approx -(1 + \delta) h_s + \\ &+ \frac{h_s}{2(1 + \delta)} [(p_x - \frac{e}{P_0} A_x)^2 + (p_y - \frac{e}{P_0} A_y)^2], \end{aligned} \quad (16)$$

where $\delta = (P - P_0)/P_0$ is the relative momentum, H_2 , p_x and p_y are normalized by the reference momentum P_0 .

Computing \dot{T}_1 only,

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} ds \frac{(1 + \frac{x}{\rho})(p_x - \frac{e}{P_0}A_x)}{1 + \delta} \approx \frac{p_x}{1 + \delta}(s_2 - s_1) \quad (17)$$

$$\begin{aligned} & p_x(s_2) - p_x(s_1) \\ &= \int_{s_1}^{s_2} ds \left\{ \frac{1 + \delta}{\rho} - \frac{(p_x - \frac{e}{P_0}A_x)^2 + (p_y - \frac{e}{P_0}A_y)^2}{2(1 + \delta)\rho} \right. \\ & \left. - \frac{(1 + \frac{x}{\rho})(-2\frac{e}{P_0}(p_x - \frac{e}{P_0}A_x)\frac{\partial A_x}{\partial x} - 2\frac{e}{P_0}(p_y - \frac{e}{P_0}A_y)\frac{\partial A_y}{\partial x})}{2(1 + \delta)} \right\} \\ & \approx -\frac{\partial}{\partial x} \int_{s_1}^{s_2} ds \frac{1}{(1 + \delta)} \left(\frac{e}{P_0}\right)^2 (A_x^2 + A_y^2) \end{aligned} \quad (18)$$

$$y(s_2) - y(s_1) = \int_{s_1}^{s_2} ds \frac{(1 + \frac{x}{\rho})(p_y - \frac{e}{P_0}A_y)}{1 + \delta} \approx \frac{p_y}{1 + \delta}(s_2 - s_1) \quad (19)$$

$$\begin{aligned} & p_y(s_2) - p_y(s_1) \\ &= -\frac{1}{2(1 + \delta)} \int_{s_1}^{s_2} ds \left(1 + \frac{x}{\rho}\right) \left(-2\frac{e}{P_0}(p_x - \frac{e}{P_0}A_x)\frac{\partial A_x}{\partial y} \right. \\ & \left. - 2\frac{e}{P_0}(p_y - \frac{e}{P_0}A_y)\frac{\partial A_y}{\partial y}\right) \\ & \approx -\frac{\partial}{\partial y} \int_{s_1}^{s_2} ds \frac{1}{(1 + \delta)} \left(\frac{e}{P_0}\right)^2 (A_x^2 + A_y^2), \end{aligned} \quad (20)$$

where we have applied the following conditions

$$\rho \Rightarrow \infty \text{ and } \int_{s_1}^{s_2} ds \int_{s_1}^s ds B_x = \int_{s_1}^{s_2} ds \int_{s_1}^s ds B_y = 0. \quad (21)$$

Note that $\rho \Rightarrow \infty$ means we use the Cartesian coordinate system. By default the reference particle gets zero bending from the insertion devices; therefore, the first integrals are required to be zero also:

$$\int_{s_1}^{s_2} ds B_y = \int_{s_1}^{s_2} ds B_x = 0. \quad (22)$$

The last line in each equation of Eqs.(17-20) repeats the results from the kick-map theory.

We would like to point out the limitations of this 1st-order theory:

1. The first-order transformation is not symplectic, even though one could make it symplectic by a “drift-kick-drift” approximation.
2. The theory is a 1st-order approximation in L ; however, there are 1st-order terms of x, y, p_x and p_y in the higher-order terms of L ; therefore, even the focusing effect is incomplete.

SUMMARY

We briefly introduced a new integration method in this short paper. We showed that the transformation in an s -dependent Hamiltonian system can be written as a series of integrals. The first order approximation reduces to the kick-map theory.

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