ELECTRON TRAJECTORIES IN A THREE-DIMENSIONAL UNDULATOR MAGNETIC FIELD*

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Abstract

It is well-known that the electron trajectory in an undulator is influenced by the focusing properties (both horizontal and vertical) of the magnetic field. The approximate solutions of motion equations for electrons in the 3-dimensional magnetic field, which describe these focusing properties, can be found by means of averaging over the short-length oscillations. On the other hand, the equations of motion can be solved numerically, by applying the Runge-Kutta algorithm. It is shown in this paper that numerically computed trajectories, which can be considered as figure of merit, on frequent occasions differ considerably from the correspondent approximate solutions obtained through the averaging method. It means that actually the undulator field influence on the electron trajectory is complicated and often cannot be reduced to the well-known focusing phenomena along.

INTRODUCTION

In the European XFEL case, long segmented planar undulators (21 segments for the SASE3 beamline to 35 for SASE1 and SASE2) are planned to be installed, with quadrupole lenses between different segments [1]. The focusing properties of undulators should be taken into account in simulations of spontaneous radiation, which constitutes the background signal of the FEL.

As far as we know horizontal and vertical focal lengths of an undulator were first calculated in [2]. In a planar undulator with infinitely wide magnetic poles and hence without horizontal focusing, the vertical focusing was analyzed in [3, 4]. In [5, 6] trajectories in the presence of focusing undulator magnetic field were calculated up to the lowest order in the initial positions and angles of the electrons. Some general relations dealing with undulator focal lengths were derived in [7 – 9]. Long-length-scale anharmonic betatron motion of electrons in very long undulators was studied in [10]. All these studies were carried out within the following limits: the focusing effects were calculated averaging over the undulator period and only terms, which are linear in the electron initial positions and angles, were taken into account.

The line width of FEL radiation is related to the dimensionless Pierce parameter ρ, which for XFEL can be as small as 3×10^{-4}. This means that the fundamental wavelength has to be tuned with an accuracy given by $\Delta\lambda/\lambda \le \rho$, whence the accuracy for undulator deflection parameter K is about of $\Delta K \approx 10^{-4}$. The expression for the phase of spontaneous radiation in the general case [11] gives the following relation for K, see Fig. 1:

$$\frac{K^2}{2} = \frac{\gamma^2}{\lambda_u}\int_0^{\lambda_u} \beta_x^2(z)dz\,. \qquad (1)$$

Here λ_u is an undulator period, γ is an electron reduced energy, β_x is its reduced horizontal velocity, being of the order of K/γ. It can be derived from (1) that the necessary accuracy for β_x is of the order of $\Delta\beta_x \approx \Delta K/\gamma \approx 3\cdot 10^{-9}$. It is doubtful whether approximate solutions for trajectories, obtained with the averaging method, provide so high accuracy.

TRAJECTORY EQUATIONS IN 3-D FIELD

We model the three-dimensional magnetic field by the following expressions which satisfy Maxwell equations:

$$B_x(x,y,z) = -\frac{k_x}{k_y}B_0\sin(k_x x)\sinh(k_y y)\sin(k_z z)\,, \qquad (2)$$

$$B_y(x,y,z) = B_0\cos(k_x x)\cosh(k_y y)\sin(k_z z)\,, \qquad (3)$$

$$B_z(x,y,z) = \frac{k_z}{k_y}B_0\cos(k_x x)\sinh(k_y y)\cos(k_z z)\,. \qquad (4)$$

Here $k_x = 1/a$, $k_z = 2\pi/\lambda_u$, $k_y = \sqrt{k_x^2 + k_z^2}$, λ_u is the undulator period length. The linear parameter a gives the field non-uniformity along X-axis and is of the order of the width of the undulator poles.

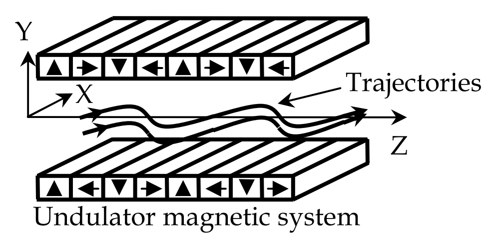

Figure 1: Sketch of a permanent magnet undulator.

We will use the exact trajectory equations in the fixed coordinate system $\{x,y,z\}$ [12]:

$$x'' = -q\sqrt{1+x'^2+y'^2}\left\{(1+x'^2)B_y - y'B_z - x'y'B_x\right\}, \qquad (5)$$

$$y'' = q\sqrt{1+x'^2+y'^2}\left\{(1+y'^2)B_x - x'B_z - x'y'B_y\right\}. \qquad (6)$$

Here β and γ are the electron's reduced velocity and energy respectively, $q = e/(mc^2\beta\gamma)$ and an apostrophe indicates a derivative with respect to z.

Substituting Eqs. (2) – (4) into Eqs. (5) and (6) and expanding them into a series about small values $k_x x$,

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$k_y y$, x' and y', we derive the following system of two nonlinear differential equations:

$$x'' = pk_z\left\{\left(1+0.5\cdot(3x'^2 - k_x^2x^2 + y'^2 + k_y^2y^2)\right)\sin\varphi - k_z yy'\cos\varphi\right\} \quad (7)$$

$$y'' = pk_z\left((k_x^2xy + x'y')\sin\varphi + k_z x'y\cos\varphi\right). \quad (8)$$

Here $K = (eB_0\lambda_u)/(2\pi mc^2)$ is an undulator deflection parameter, $p = K/(\beta\gamma)$ and $\varphi = k_z z$.

The change-over from precise equations (5) and (6) to their approximations (7) and (8) must be accompanied by the accuracy loss. To estimate it, a computer code was written, which solves the systems of equations (5, 6) and (7, 8) by using the Runge-Kutta algorithm. The simulations were performed with the European XFEL parameters listed in the Table 1 (see [1]).

Table 1: European XFEL Parameters for Simulation

Electron beam energy	17.5 GeV
Normalized emittance ε_n	10^{-6} m rad
RMS beam emittance $\varepsilon = \varepsilon_n/\gamma$	2.9*10^{-11} m rad
Averaged beta-function	15 m
Undulator period λ_u	40 mm
Undulator deflection parameter K	4
Horizontal non-uniformity a	50 mm
Number of undulator periods N	124
Initial transversal position x_0, y_0	0,03 mm
Initial deflection $\theta_0 = x_0' + K/\gamma, y_0'$	2*10^{-3} mrad

The computed results are presented in Table 2. Here $\Delta x = \max\left|x_{56}(z) - x_{78}(z)\right|$, where $x_{56}(z)$ is the solution of differential Eqs. (5) and (6) computed numerically, and $x_{78}(z)$ is the corresponding solution of Eqs. (7) and (8). Similar notation is understood for Δy, $\Delta x'$ and $\Delta y'$.

Table 2: Differences in Solutions of Eqs. (5, 6) and (7, 8)

	ε_n	$100\varepsilon_n$
Δx, mm	2.6*10^{-11}	1.2*10^{-7}
Δy, mm	3.0*10^{-10}	3.0*10^{-7}
$\Delta x'$	1.6*10^{-14}	1.0*10^{-10}
$\Delta y'$	1.4*10^{-13}	1.5*10^{-10}

The column ε_n corresponds to initial coordinates and deflections listed in Table 1, and the column $100\varepsilon_n$ corresponds to initial values which are increased in 10 times each. It is clear that differences are small, and that the approximate differential equations (7) and (8) can be safety used for trajectory simulations.

FOCUSING EFECTS

By neglecting all small terms in equations (7) and (8), which are quadratic in x, x', y and y', we get the electron trajectory in linear approximation:

$$x_1(z) = x_0 + \theta_0 z - (p/k_z)\sin(\varphi), \quad (9)$$

$$y_1(z) = y_0 + y_0'z. \quad (10)$$

We may generalize these expressions, searching for “focusing” solutions for equations (7) and (8) in the form:

$$x_f(z) = x_s(z) - (p/k_z)\sin(\varphi), \quad (11)$$

$$y_f(z) = y_s(z), \quad (12)$$

where $x_s(z)$ and $y_s(z)$ are slowly varying functions. Substituting equations (11) and (12) into (7) and (8), and averaging over the undulator period (thus cancelling the fast oscillating terms), we get the following equations for the slowly varying components of the trajectory [5, 6]:

$$x_s''(z) - k_z^2\omega_x^2 x_s(z) = 0, \quad (13)$$

$$y_s''(z) + k_z^2\omega_y^2 y_s(z) = 0, \quad (14)$$

where $\omega_x = (pk_x)/(\sqrt{2}k_z)$ and $\omega_y = (pk_y)/(\sqrt{2}k_z)$ are the dimensionless betatron oscillations periods in units of λ_u along the horizontal and the vertical directions. Solving these equations, we get the following expressions for $x_f(z)$ and $y_f(z)$:

$$x_f(z) = x_0\cosh(\omega_x\varphi) + \frac{\theta_0}{\omega_x k_z}\sinh(\omega_x\varphi) - \frac{p}{k_z}\sin(\varphi), \quad (15)$$

$$y_f(z) = y_0\cos(\omega_y\varphi) + \frac{y_0'}{\omega_y k_z}\sin(\omega_y\varphi), \quad (16)$$

where $\theta_0 = x_0' + p$.

Below we will consider relatively short undulators, with the number of periods N such that:

$$2\pi N\omega_{x,y} << 1 \quad (17)$$

In the case of the European XFEL conditions (17) are clearly fulfilled as $N = 124$, $k_{x,y} \sim k_z$, $\omega_{x,y} \sim p \cong 10^{-4}$.

Differentiating expressions (15) and (16) with respect to z, we get for relatively short undulators:

$$x_f'(z) = x_0\omega_x^2 k_z\varphi + \theta_0(1 + 0.5\omega_x^2\varphi^2) - p\cos(\varphi) \quad (18)$$

$$y_f'(z) = -y_0\omega_y^2 k_z\varphi + y_0'(1 - 0.5\omega_y^2\varphi^2) \quad (19)$$

The horizontal focal length f_x can be deduced from the geometrically evident relation: $\frac{1}{f_x} = -\frac{dx'(z = N\lambda_u)}{dx_0}$.

As a result for f_x we obtain [2]: $1/f_x = -2\pi Nk_z\omega_x^2$.

The relation $1/f_y = 2\pi Nk_z\omega_y^2$ can be derived similarly.

Although relations (15) and (16) have been universally accepted, in some practically important cases they may be in sharp contrast with more accurately simulated results.

NUMERICAL SIMULATIONS

Let us compare the numerically computed trajectory (marked as "RK" – Runge-Kutta) with those given by eqs. (15), (16) (marked as "focus"). A computer code was written for accurate simulation of trajectories in undulator field, which employs the Runge-Kutta algorithm. The data from Table 1 were used for simulations. For simplicity, here we extracted the functions x_1 and y_1 (see (9), (10)) from trajectories:

$$\Delta X_{RK} = x_{RK}(z) - x_1(z)\,,\ \Delta X'_{RK} = x'_{RK}(z) - x'_1(z)\,,$$

$$\Delta X_{focus} = x_f(z) - x_1(z)\,,\ \Delta X'_{focus} = x'_f(z) - x'_1(z)\,.$$

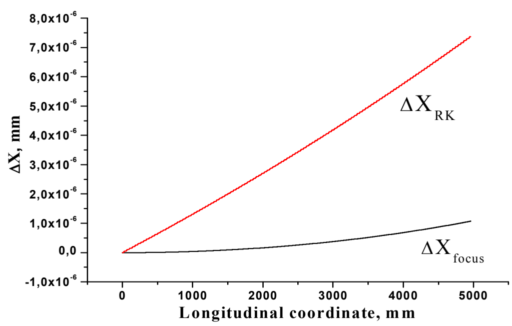

Figure 2. Focusing vs. numerically simulated components of horizontal coordinate.

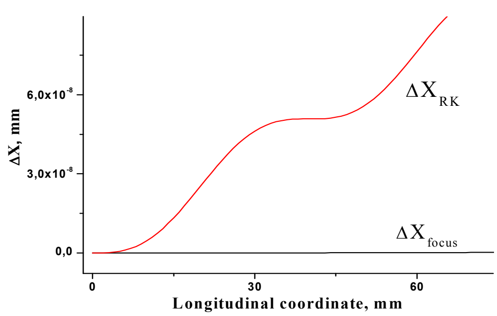

Figure 3. The same as Fig. 2, starting part of trajectory.

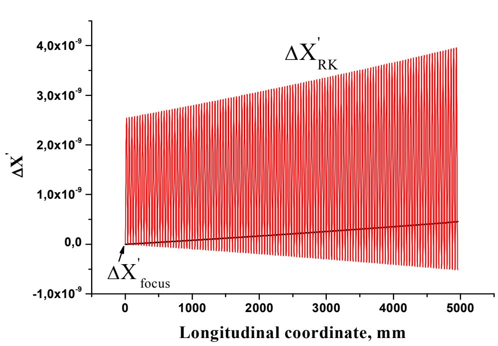

Figure 4. Focusing vs. numerically simulated components of horizontal velocity.

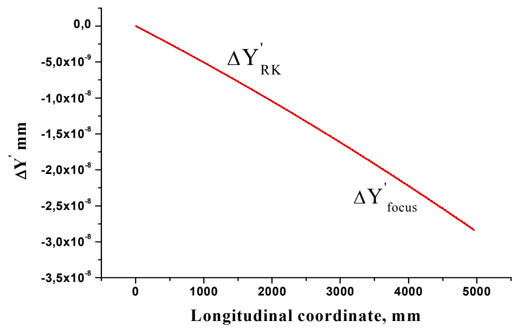

Figure 5. Focusing vs. numerically simulated components of vertical velocity.

This example of calculations demonstrates that the influence of transverse non-uniformities of the undulator field on the horizontal component of the electron trajectory is considerably larger than what follows from simple relations (15) and (16). At the same time, the influence on the vertical component of trajectory is relatively small.

Notice that approximate analytical solutions for Eqs. (7) and (8) can be derived [13], which are much more accurate compared with Eqs. (15), (16). These analytical solutions also confirm the results presented here.

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