

NUMERICAL SIMULATIONS OF TRANSVERSE MODES IN GAUSSIAN BUNCHES WITH SPACE CHARGE*

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Abstract

The transverse modes and the intrinsic Landau damping in Gaussian bunched beams with space charge are numerically investigated. The evolution of the phase space density is calculated with the Synergia accelerator modeling package and analyzed with Dynamic Mode Decomposition (DMD) method. DMD is a relatively new technique used to calculate mode dynamics in both linear and nonlinear systems. The properties of the first three space charge modes, including their shape, damping rates and tune shifts are calculated over the entire range of the space charge interaction.

INTRODUCTION

Landau damping provides an important mechanism for stabilizing beam propagation. The mechanism operates via an energy exchange between the potentially unstable coherent mode and some resonant particles. A necessary condition for the Landau damping mechanism is the existence of resonant particles with a continuous frequency spectrum around the coherent frequency. In accelerators, aside from the tune spread due to nonlinear lattice elements, there is a tune spread caused by the space charge interaction which plays an important and not-well-understood role in the damping mechanism. Here we neglect the nonlinear lattice effects and focus only on the intrinsic effect of space charge on the transverse modes of a Gaussian bunch. The beam dynamics are investigated over the entire range of the interaction, from no space charge to the strong space charge limit.

The effect of space charge on the head-tail modes was first addressed by Blaskiewicz [1] who showed that space charge can improve beam stability. The strong space charge regime for bunches of arbitrary shape was addressed analytically in [2, 3]. The mode shapes, tune shifts and damping rates were calculated for Gaussian bunches. A numerical investigation of the transverse modes in longitudinal Gaussian bunches with K-V transverse distribution using particle tracking simulations was done by V. Kornilov *et al.* [4]. Our simulations for bunches with K-V charge distribution (not shown here) are in agreement with the ones in Ref [4].

ANALYTICAL RESULTS

The tune shift for a particle propagating through a lattice and experiencing a transverse space charge force $\frac{eE_x}{\gamma^2}$ is [5]

$$\Delta Q_{sc} = \frac{e}{m\gamma^3\beta^2c^2} \frac{1}{4\pi} \oint \beta(s) \frac{\partial E_x}{\partial x}(\bar{x}, s) ds. \quad (1)$$

where $\beta(s)$ is the lattice beta function.

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We define the transverse displacement density as

$$X(z, u, s) = \frac{\int dx dp_x dy dp_y x \rho(x, p_x, y, p_y, z, u, s)}{\rho(z, u)} \quad (2)$$

where $\rho(x, p_x, y, p_y, z, u, s)$ is the density in phase space with z being the longitudinal position relative to the reference particle and $u = \frac{\delta p}{p}$ being the relative momentum spread. Following Ref [3] and defining $X(z, u)$ such as

$$X(z, u, s) = X(z, u) e^{(-i\omega_0 Q_s - i\chi z)} \quad (3)$$

where ω_0 is the reference particle angular velocity and χ is the effective chromaticity one gets the equation of motion

$$(Q - Q_\beta)X(z, u) + iQ_s \frac{\partial X}{\partial \theta}(z, u) = -\Delta Q_{sc \text{ eff}}(z)(X(z, u) - \bar{X}(z)), \quad (4)$$

with θ being the synchrotron oscillation phase. $\Delta Q_{sc \text{ eff}}(z)$ is the effective space charge tune shift. For a Gaussian beam we define the space charge parameter as

$$q_{\text{eff}} = \frac{\Delta Q_{sc \text{ eff max}}}{Q_s} = 0.52 \frac{\Delta Q_{sc \text{ max}}}{Q_s} \quad (5)$$

where $\Delta Q_{sc \text{ max}}$ is the maximum tune shift (at the beam center) and Q_s is the synchrotron tune. This definition is in agreement with the one in Ref [2]. For a Gaussian beam the effective tune shift is about 0.52 smaller than the maximum tune shift [2]. In order to determine q_{eff} we numerically calculate the integral in Eq. 1 using the electric field from the simulation.

When the space charge is zero there is a simple solution to Eq. 4,

$$X(z, u) \equiv X(r, \theta) = R(r) e^{im\theta}, \quad (6)$$

with r and θ being the amplitude and the phase of the synchrotron oscillation. The tune shift is given by

$$Q - Q_\beta = mQ_s \quad (7)$$

for integer m . The modes are defined by the angular number m and are radially degenerate.

Large q_{eff} requires solutions which are weakly dependent on u , i.e. $X(z, u) \approx X(z)$. A detailed calculation of the modes properties is presented in [2, 3]. The strong space charge modes form an orthogonal set

$$\int X_k(z) X_m(z) \rho(z) dz = \delta_{km}, \quad (8)$$

where k and m represent mode numbers. For the mode k the tune shift is $\frac{\nu_k Q_s}{q_{\text{eff}}}$ (the values of ν_k are tabulated in [2]),

while the Landau damping is $\lambda_k = \frac{k^4 2\pi Q_s}{q_{\text{eff}}^3}$.

SIMULATION

Dynamic Mode Decomposition (DMD) is a data-driven algorithm used for modal analysis and model reduction in both linear and nonlinear systems [6–8]. Developed initially by the fluid mechanics community to address turbulent flows, it has proven a successful tool in many different areas [6–10]. One important feature of this method is the direct calculation of mode dynamics, including mode shape, frequency and growing/damping rates. The algorithm description and implementation for beam modes analysis can be found elsewhere [11].

For this study we employ the Synergia accelerator modeling package [12, 13] to simulate the propagation of a single Gaussian beam through a linear lattice consisting of drift, quadrupole and rf cavity elements. The single particle dynamics is purely linear. The chromaticity is zero and the longitudinal potential is parabolic. Space charge is calculated with a 3D Poisson solver with open boundary conditions [14]. The transverse displacement density, $X(z, u, s)$, is stored every turn and analyzed using Dynamic Mode Decomposition.

RESULTS

The zero mode corresponds to the transverse displacement of the beam as a whole, i.e. $X(z, u)$ is constant. In our model which neglects the wake fields and lattice nonlinearities its dynamics is described by a simple betatron oscillation.

To validate our simulations we start in the strong space charge regime where comparison with theoretical results can be done. In agreement with the theory [2, 3] we find that the modes' spatial distribution is nearly independent of the momentum coordinate u . In Fig. 1 we compare the first four modes dependence on z with the theoretical shape calculated in Ref [2]. The agreement is very good.

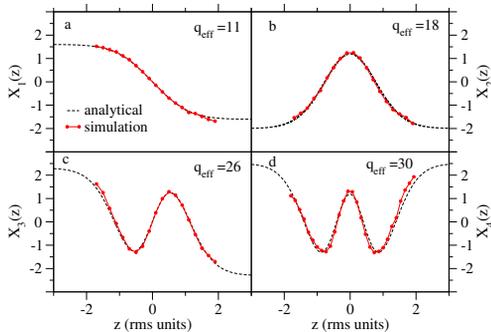


Figure 1: Comparison of the simulated modes' shape with the theoretical one [2] in the strong space charge regime. The agreement is good.

In order to study the evolution of the strong space charge modes into the weakly interacting regime we excite the beam with excitations proportional to the shape of the strong space charge modes. Among the many excited modes we choose for investigation the ones which have the largest overlap with

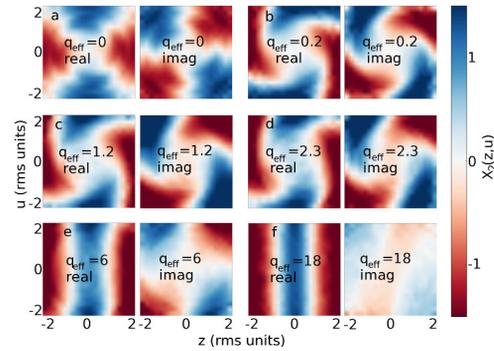


Figure 2: The mode 2 longitudinal distribution, $X_2(z, u)$, for different values of the space charge parameter q_{eff} . Without space charge, $X_2(z, u) \propto e^{i2\theta}$. With increasing q_{eff} , $X_2(z, u)$ transforms gradually to the strong space charge mode 2. At large q_{eff} $X_2(z, u)$ can be described by a purely real function.

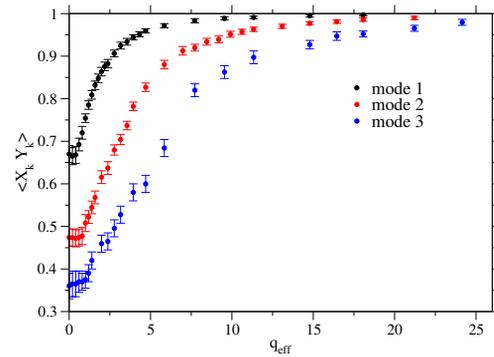


Figure 3: The overlap $\langle X_k Y_k \rangle$ (see Eq. 9) in the longitudinal phase space of the mode k with the strong space charge space mode k for $k = 1, 2, 3$. At small q_{eff} the overlap increases rapidly with increasing q_{eff} . For $q > q_{ck}$ ($q_{c1} \approx 4$, $q_{c2} \approx 7$, $q_{c3} \approx 9$) the increase of $\langle X_k Y_k \rangle$ with q_{eff} saturates indicating the transition to the strong space charge regime.

the strong space charge modes. These are also the modes which changes smoothly when going from large to small q_{eff} .

We calculated the first three modes spatial distribution in the longitudinal phase space for different values of q_{eff} . When $q_{eff} = 0$ the modes are characterized by the angular number $m = k$ (where k is the mode number), i.e. $X_k(z, u) \equiv X_k(r, \theta) = R_k(r)e^{ik\theta}$. With increasing q_{eff} the modes transform gradually into the strong space charge modes. While at small q_{eff} the real and the imaginary part of the modes have comparable magnitudes, at large q_{eff} the modes can be described by purely real functions. For illustration we show in Fig. 2 the mode 2 shape evolution from the noninteracting to the strong space charge regime.

The spatial overlap of the mode $X_k(z, u)$ with the strong space charge mode $Y_k(z)$ (i.e. $Y_k \equiv X_k(q_{eff} = \infty)$),

$$\langle X_k Y_k \rangle = \int X_k^*(z, u) Y_k(z) \rho(z, u) dz du, \quad (9)$$

as a function of q_{eff} is presented in Fig. 3. At small q_{eff} the overlap increases rapidly with increasing q_{eff} . At large q_{eff}

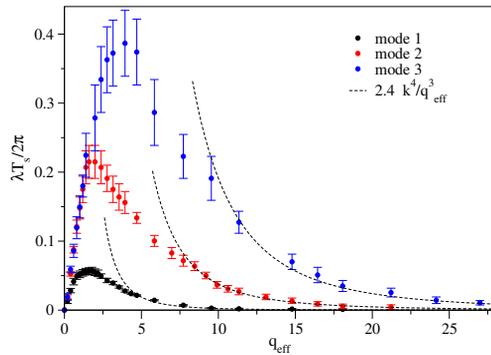


Figure 4: The Landau damping for modes 1, 2 and 3 versus the space charge parameter q_{eff} . T_s is the synchrotron period. At small q_{eff} the damping increases quickly with increasing q_{eff} . In the strong space charge regime we find that $\lambda \approx 2.4 \frac{k^4 2\pi Q_s}{q_{eff}^3}$, where k is the mode number (dashed lines). Up to a proportionality factor which we find to be sensitive to beam transverse distribution, this behavior is in agreement with the theoretical predictions of Ref. [2]. The proportionality factor of 2.4 is characteristic to transverse Gaussian beams.

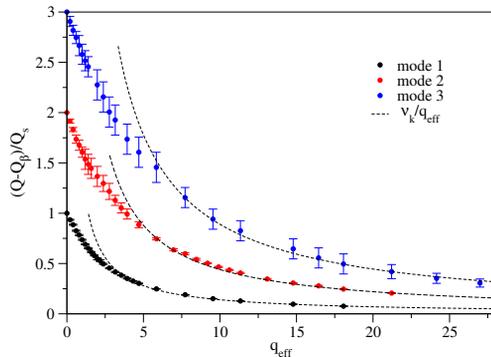


Figure 5: The relative tune shift, $\frac{Q-Q_\beta}{Q_s}$, for the modes 1, 2 and 3 versus the space charge parameter q_{eff} . In the strong space charge regime, $q > q_{ck}$, $\frac{\Delta Q}{Q_s} \approx \frac{v_k}{q_{eff}}$, in good agreement with the theoretical prediction [2] (dashed lines). The values of the constants v_k are tabulated in Ref. [2].

the overlap slope is much smaller. For every mode we can determine the space charge parameter q_{ck} where the increase of $\langle X_k Y_k \rangle$ with q_{eff} shows signs of saturation and define it as the point where the transition to the strong space charge regime takes place. An estimate for the first three modes is $q_{c1} \approx 4$, $q_{c2} \approx 7$ and $q_{c3} \approx 9$. These values are consistent with the estimates based on damping and tune shift behavior, as shown below.

The Landau damping for the first three modes is shown in Fig. 4. All three modes display a similar behavior. At small q_{eff} the damping increases quickly with increasing q_{eff} . The damping reaches its maximum in the intermediate region. The maximum damping and the corresponding q_{eff} increases with the mode number. For $q_{eff} > q_{ck}$ the damping

$\lambda \approx 2.4 \frac{k^4 2\pi Q_s}{q_{eff}^3}$. This dependence is in good agreement with the theoretical predictions of Ref [2] up to a proportionality factor. As simulations with K-V beams shows (not shown), the proportionality factor is sensitive to the transverse bunch distribution. The value of 2.4 is characteristic to transverse Gaussian beams.

In Fig. 5 the tune shifts for the first three modes as a function of q_{eff} are shown. Starting from the non-interacting value of $Q = Q_\beta + kQ_s$, the tune is suppressed with increasing q_{eff} . In the strong space charge regime (i.e. $q > q_{ck}$) $\frac{Q-Q_\beta}{Q_s} \approx \frac{v_k}{q_{eff}}$ in good agreement with the theoretical estimate [2]. v_k are constants coefficients specific to each mode and were calculated in Ref. [2].

CONCLUSIONS

A numerical investigation of the transverse modes and intrinsic Landau damping in Gaussian bunched beams with space charge is presented. The phase space density and the transverse displacement density are simulated using the Synergia accelerator simulation package. The Dynamic Mode Decomposition technique is used to extract the modes' properties from the simulated data.

As the strength of the space charge is increased, the modes' shapes change gradually from the radially degenerate non-interacting modes to the momentum-independent space charge modes predicted by the theory. Based on the evolution of the modes shape, damping and tune shift with increasing q_{eff} , we estimate $q_{c1} \approx 4$, $q_{c2} \approx 7$ and $q_{c3} \approx 9$ as threshold values which define the strong space charge regime for the first three modes.

In the strong space charge regime the damping for the Gaussian beams is $\approx 2.4 \frac{k^4 2\pi Q_s}{q_{eff}^3}$. The modes' tune shift is in agreement with the theoretical predictions and equal to $\frac{v_k Q_s}{q_{eff}}$, where v_k constants are tabulated in Ref. [2].

The damping reaches maximum in the intermediate region. Both the maximum damping and the corresponding value of q_{eff} increase with the mode number. The space charge mode k connects with a non-interacting mode with the angular number $m = k$ upon decreasing q_{eff} .

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