ISBN: 978-3-95450-168-7 doi:10.18429/JACoW-IPAC2015-MOPMA058

EFFECT OF SPHERICAL ABERRATION ON BEAM EMITTANCE GROWTH*

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Abstract

Spherical aberration in axially symmetric magnetic focusing lenses results in an S-shape figure of beam emittance. Filamentation of beam emittance in phase space is a fundamental property of a beam affected by aberrations. An analytical expression for effective beam emittance growth due to spherical aberration is obtained. Analysis is extended for intial emittance growth in drift space and within a focusing channel, induced by beam space charge.

BEAM EMITTANCE GROWTH IN AXIAL-SYMMETRIC LENS

Beam dynamics in an axially-symmetric field may be severely affected by spherical aberrations, which results in beam emittance growth. Magnetic field of the focusing lens can be approximated as

$$B(z) = \frac{B_o}{1+(z/d)^n} \ , \qquad (1)$$

where B_o is the peak field, and d is the characteristic length. Parameter n = 2 corresponds to the well-known Glazer model [1] for short length/diameter lenses, while in many cases the field of a solenoid is better approximated by n = 4, which yields a flatter distribution.

Let us estimate the emittance growth of a beam during its passage through the lens. We assume that the position of the particle is not changed while crossing the lens, and only the slope of the particle trajectory is altered. The transformation of particle variables before $(r_o,\ r_o')$ to after $(r,\ r')$ lens-crossing is given by [2]:

$$r = r_o \ , \qquad r' = r_o' - \frac{r}{f}(1 + C_\alpha r^2)\ , \qquad (2)$$

where $f = (4/D)(mc\beta\gamma / qB_o)^2$ is the focal length of the lens, D is the effective length of the lens, and C_α is the spherical aberration coefficient. For the Glazer model $D = (\pi/2)d$, $C_\alpha = 0.25 d^{-2}$, and for $n = 4$ $D = d3\pi/4\sqrt{2}$, $C_\alpha = (5/12)d^{-2}$ [3]. In many applications, the spherical aberration coefficient can be expressed through solenoid sizes as $C_\alpha = 5/(S+2a)^2$, where $2a$ is the pole piece diameter and S is the solenoid pole gap width [4].

Suppose that the initial phase space volume of the beam with radius R is bounded by the ellipse

$$\frac{r_o^2}{R^2}\ni + \frac{r_o'^2}{\ni}R^2 = \ni\ , \qquad (3)$$

where $\ni$ is the unnormalized beam emittance. To find the deformation of the boundary of the beam phase space after passing through the lens, let us substitute the inverse transformation $r_o = r$, $r_o' = r' + (r/f)(1 + C_\alpha r^2)$ into the ellipse equation, Eq. (3). The boundary of the new phase space volume, occupied by the beam after passing through the lens at phase plane $(r,\ r')$ is given by:

$$\frac{r^2}{R^2}\ni + \frac{R^2}{\ni}\left(r' + \frac{r}{f} + C_\alpha \frac{r^3}{f}\right)^2 = \ni\ . \qquad (4)$$

Let us introduce new variables $(T,\ \psi)$ arising from the transformation:

$$\frac{r}{R} = \sqrt{T}\cos\psi\ , \qquad \left(r' + \frac{r}{f}\right)\frac{R}{\ni} = \sqrt{T}\sin\psi\ . \qquad (5)$$

In terms of the new variables, the shape of the beam emittance after lens-crossing is

$$T + 2\upsilon T^2 \sin\psi\cos^3\psi + T^3\upsilon^2\cos^6\psi = 1\ , \qquad (6)$$

where the following notation is used

$$\upsilon = \frac{C_\alpha R^4}{f\ni}\ . \qquad (7)$$

Without nonlinear perturbations, $\upsilon = 0$, and Eq. (6) describes a circle in phase space. Conversely, if $\upsilon \neq 0$, Eq. (6) describes an S – shape distorted figure of beam emittance (see Fig. 1). Accordingly, filamentation of beam emittance in phase space is a fundamental property of a beam affected by aberrations.

Being symplectic in nature, the transformation, Eq. (2), conserves phase-space area. However, the effective area occupied by the beam, increases as a result of the encounter. Let us determine the increase in effective beam emittance as a square of the product of minimum and maximum values of T:

$$\frac{\ni_{eff}}{\ni} = \sqrt{T_{min}T_{max}}\ . \qquad (8)$$

The values T_{max}, T_{min} are determined numerically from Eq. (6). Dependence of the emittance growth on the parameter υ is shown in Fig. 2.

Qualitatively, this relationship can be approximated by the function:

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Table 1. Coefficients in Emittance Growth for Different Beam Distributions.

Distribution	K Eq. (10)	$\bar{K}$ Eq. (15)	$\Delta W / W_o$ Eq. (19)
KV	0.0556	0	0
Water Bag (WB)	0.114	0.094	0.01126
Parabolic (PB)	0.164	0.187	0.02366
Gaussian (GS)	0.541	0.55	0.077

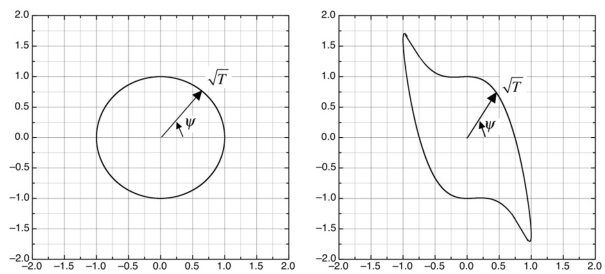

Figure 1: Distortion of beam emittance due to spherical aberrations, Eq. (6): (left) $\upsilon = 0$, (right) $\upsilon = 1.6$.

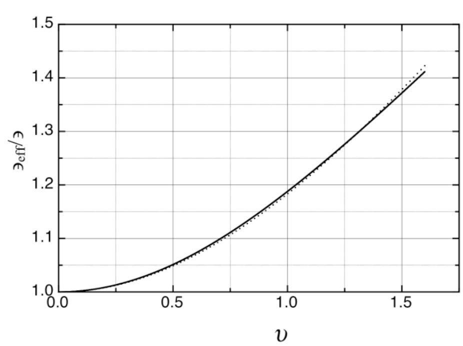

Figure 2: Effective beam emittance as a function of parameter υ, Eq. (7): (solid) Eq. (8), (dotted) Eq. (9).

$$\frac{\epsilon_{eff}}{\epsilon} = \sqrt{1 + K\upsilon^2} \ , \qquad (9)$$

where the parameter $K \approx 0.4$. Substitution of Eq. (7) into Eq. (9) gives the following expression for effective beam emittance growth:

$$\frac{\epsilon_{eff}}{\epsilon} = \sqrt{1 + K(\frac{C_\alpha R^4}{f\,\epsilon})^2} \ . \qquad (10)$$

Equation (10) was tested numerically for a round beam with different particle distributions. Simulations were performed with macroparticle code BEAMPATH. As a measure of effective beam emittance, the four-rms beam emittance $\epsilon = 4\sqrt{<x^2><x'^2> - <xx'>^2}$ was used and a two-rms beam size, $R = 2\sqrt{<x^2>}$, was used as a measure of the beam radius. Simulations confirm that the dependence given by Eq. (10), is valid for four-rms beam emittance, although the coefficient K depends on the beam distribution (see Table 1). Fig. 3 illustrates beam emittance growth of the beam with a Gaussian distribution due to spherical aberration.

SPACE CHARGE INDUCED BEAM EMITANCE GROWTH IN DRIFT SPACE

Non-linear space charge forces inherent to a non-uniform beam act on the beam as a non-linear lens. The developed analysis can be applied to determine space-charge induced beam emittance growth. Consider the initial stage of beam drift in free space at a certain distance *z,* where radial particle positions are not changed significantly, but the momentum distribution is already affected by the space charge field of the beam, $E_b(r)$. Change in the radial slope of the particle trajectory is given by

$$r' = r_o' + \frac{qzE_b(r)}{mc^2\beta^2\gamma^3} \ . \qquad (11)$$

Consider a Gaussian beam in drift space. The space charge field of the beam is approximated as:

$$E_b(r) = I\,\frac{1\text{-}\exp(-2r^2/R^2)}{2\pi\varepsilon_o\beta c r} \approx \frac{I}{\pi\varepsilon_o\beta c}\frac{r}{R^2}(1\text{-}\frac{r^2}{R^2}+..) \ , \qquad (12)$$

where I is the beam current. Substitution of Eq. (12), into Eq. (11) results in a transformation, Eq. (2), where

$$\frac{1}{f_b} = -\,4\frac{I}{I_c(\beta\gamma)^3}\frac{z}{R^2} \ , \qquad C_\alpha = -\frac{1}{R^2} \ , \qquad (13)$$

and $I_c = 4\pi\varepsilon_o mc^3/q$ = $3.13\text{x}10^7 A/Z$ [Amp] is the characteristic beam current. Parameter υ, Eq. (7), which determines the effect of spherical aberration on the beam emittance is therefore

$$\upsilon = \frac{C_\alpha R^4}{f_b\,\epsilon} = \frac{4}{\beta^3\gamma^3}\frac{I}{I_c}\frac{z}{\epsilon} \ . \qquad (14)$$

Substitution of Eq. (14) into Eq. (9) results in the following expression for the initial beam emittance growth in free space due to space charge:

$$\frac{\epsilon_{eff}}{\epsilon} = \sqrt{1 + \bar{K}(\frac{I}{I_c\beta^3\gamma^3}\frac{z}{\epsilon})^2} \ . \qquad (15)$$

The parameter $\bar{K}$ was determined numerically for different distributions (see Table 1). As follows from Eq. (15), initial emittance growth does not depend on initial beam radius. The same result was obtained in Ref. [5] for emittance growth of the beam with an initial waterbag distribution and initial zero emittance in free space, defined as $\epsilon_{eff} = 0.298\,zI/(I_c\beta^3\gamma^3)$. This expression coincides with Eq. (15) for $\epsilon = 0$, $\bar{K} = 0.094$.

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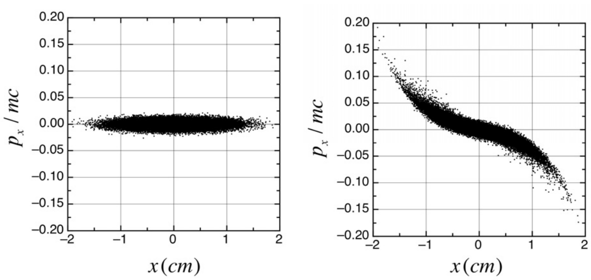

Figure 3: Emittance growth of the beam with Gaussian distribution in a lens with $\upsilon = 1.6$, Eq. (7).

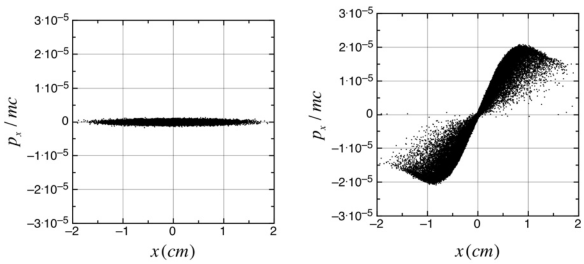

Figure 4: Emittance growth of a 150 keV proton beam with initial Gaussian distribution, current I = 1.11mA and beam emittance $\varepsilon = 0.039\ \pi$ cm mrad in drift space of length z = 100 cm.

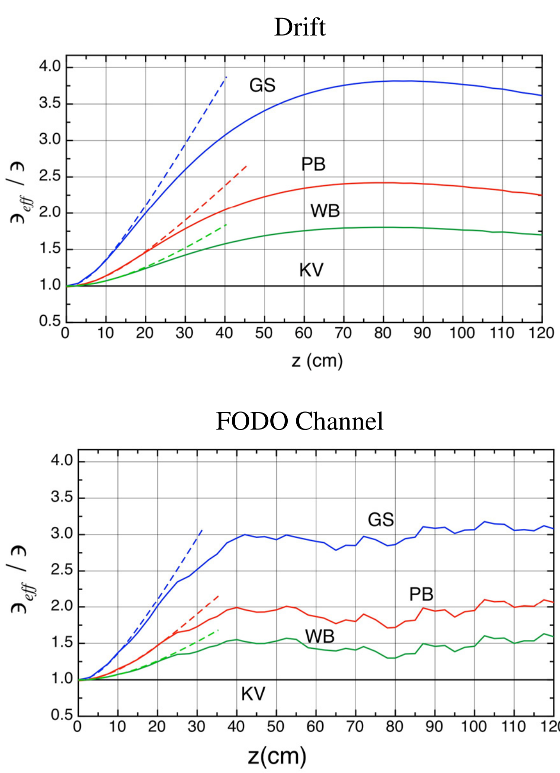

Figure 5: Emittance growth of a 50 keV proton beam with current I = 20 mA and emittance $\varepsilon = 4.64\ \pi$ cm mrad in drift space and in FODO focusing channel for different distribution (see Table 1): (solid) numerical solution, (dotted) initial emittance growth determined by Eq. (15).

BEAM EMITANCE GROWTH IN FOCUSING CHANNEL

Within the context of the smooth approximation, the radial equation of motion of a particle in combination of linear focusing field and space charge field is

$$\frac{d^2 r}{dz^2} = -\frac{\mu_o^2}{L^2} r + \frac{qE_b(r)}{mc^2 \beta^2 \gamma^3}, \qquad (16)$$

where μ_o is the phase advance of transverse oscillations in a focusing channel per period L of the focusing structure. In the beginning of beam transport, the change of slope of particle trajectory can be represented as a combination of a kick arising from the focusing field, and that arising from the space charge field:

$$r' = r_o' - \frac{r}{f_{ext}} - \frac{r}{f_b}(1 + C_\alpha r^2), \qquad (17)$$

where $f_{ext} = L^2 / (\mu_o z)$. Eq. (17) can be rewritten as Eq. (2) with an effective focal length $\bar{f}$, and effective aberration coefficient $\bar{C}_\alpha$ defined as

$$\frac{1}{\bar{f}} = \frac{f_b + f_{ext}}{f_b f_{ext}}, \qquad \bar{C}_\alpha = C_\alpha \frac{f_{ext}}{f_b + f_{ext}}. \qquad (18)$$

From Eq. (18), the ratio of effective aberration coefficient to effective focal length is the same as that for the beam in drift space, $C_\alpha / f_b = \bar{C}_\alpha / \bar{f}$. Therefore, initial emittance growth in focusing channel is determined by the same Eq. (15). Final emittance growth of an rms-matched beam in a focusing channel is limited by beam intensity redistribution ("free energy" effect) [6]:

$$\frac{\varepsilon_f}{\varepsilon} = \sqrt{1 + \frac{2}{(\beta\gamma)^3} \frac{I}{I_c} \left(\frac{R}{\varepsilon}\right)^2 \left(\frac{\Delta W}{W_o}\right)}, \qquad (19)$$

where $\Delta W / W_o$ is the free energy coefficient, depending on the initial beam distribution (see Table 1). Emittance growth in focusing channel takes place at the distance equivalent to quarter of plasma oscillation amplitude of an infinitely large beam [6]:

$$z = \frac{\pi}{4} R \sqrt{\frac{I_c}{I} (\beta\gamma)^3}. \qquad (20)$$

Figure 5 illustrates beam emittance growth of a proton beam in drift space as well as in a FODO focusing channel. As seen, Eq. (15) gives good approximation of emittance growth at the initial stage.

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