ISBN: 978-3-95450-168-7 doi:10.18429/JACoW-IPAC2015-MOPMN019

UNDERSTANDING THE EFFECT OF SPACE CHARGE ON INSTABILIIES*

M. Blaskiewicz†, BNL, Upton NY 11973 USA
A. Chao, SLAC, Stanford University, Stanford CA 94309 USA
Y.H. Chin, KEK, Ibaraki, Japan

Abstract

The combined effect of space charge and wall impedance on transverse instabilities is an important consideration in the design and operation of high intensity hadron machines as well as an intrinsic academic interest. This study explores the combined effects of space charge and wall impedance using various simplified models in an attempt to produce a better understanding of their interplay.

TWO PARTICLE MODEL

The simplest nontrivial model including the space charge force requires two macroparticles [1, 2]. We consider vertical motion and use $\theta = s/R$ as the dynamical variable, it increases by 2π each turn. When particle two leads particle one during the synchrotron oscillation one has:

$$y_1'' = -\nu^2 y_1 + K(y_1 - y_2) + W y_2 \tag{1}$$
$$y_2'' = -\nu^2 y_2 + K(y_2 - y_1), \tag{2}$$

where ν is the unperturbed betatron tune, $'$ denotes $d/d\theta$, K creates the space charge tune shift and W is the wake strength. Indicies are reversed during the second half of the synchrotron oscillation. To proceed we use the single sideband approximation. Assume

$$y_i(\theta) = \hat{y}_i(\theta)\exp(-iA\theta),$$

insert these in eq (1) and(2), and neglect terms proportional $\hat{y}_i''$. Setting $A^2 = \nu^2 - K$ gives

$$\hat{y}_1' = -i\frac{K-W}{2A}\hat{y}_2 \tag{3}$$
$$\hat{y}_2' = -i\frac{K}{2A}\hat{y}_1 \tag{4}$$

Set $\kappa = \sqrt{(K-W)/K}$ and $b_0 = K\kappa/2A$ so that

$$\hat{y}_1(\theta) = \cos(b_0\theta)\hat{y}_{10} - i\kappa\sin(b_0\theta)\hat{y}_{20} \tag{5}$$
$$\hat{y}_2(\theta) = \cos(b_0\theta)\hat{y}_{20} - \frac{i}{\kappa}\sin(b_0\theta)\hat{y}_{10}, \tag{6}$$

where $\hat{y}_{10}$ and $\hat{y}_{20}$ are intial conditions.When b_0 is imaginary we use $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$. Setting $\theta = \pi/\nu_s$ with ν_s the synchrotron tune yields the map for the first half of the synchrotron oscillation. Reversing the roles of particle 2 and 1 yields the map for the second half. Concatenating the maps yields a two by two matrix with unit determinant. The trace of the full matrix is

$$Tr(M_0) = 2\cos^2(b_0\pi/\nu_s) - \left\{\kappa^2 + 1/\kappa^2\right\}\sin^2(b_0\pi/\nu_s). \tag{7}$$

Let λ be an eigenvalue of M. Then $Tr(M) = \lambda + 1/\lambda$ and the system is unstable if $|Tr(M)| > 2$.

AN ALTERNATE SOLUTION

By neglecting the terms proportional to $\hat{y}''$ in the previous section we introduced errors in the tunes appearing in the transport matrix. These can be avoided without sacrificing a simple closed form solution. We start off by diagonalizing equations (1) and (2). Set $z = y_1 + \alpha y_2$, where α is an unknown constant. One gets

$$z'' = -A^2 z - \alpha K\kappa y_1 - (K-W)y_2.$$

Now introduce another unknown constant β and demand $\beta z = \alpha K\kappa y_1 - (K-W)y_2$. This gives $\alpha = \pm\kappa$ and $\beta = \alpha K$.The new equations of motion are

$$z_1'' = -(A^2 + K\kappa)z_1 \qquad z_2'' = -(A^2 - K\kappa)z_2 \tag{8}$$

with $z_1 = y_1 + \kappa y_2$ and $z_2 = y_1 - \kappa y_2$. Define

$$B_1 = \sqrt{A^2 + K\kappa}, \qquad B_2 = \sqrt{A^2 - K\kappa}.$$

Now, assume κ is real and positive along with the Bs. We deal with imaginary κ later. Approximate

$$\hat{z}_m(\theta) \equiv z_m(\theta) + iz_m'(\theta)/A = \hat{z}_m(0)e^{-iB_m\theta}, \tag{9}$$

where we would divide by B_m instead of A for an exact solution. To the same level of approximation one has

$$\hat{y}_1 = (\hat{z}_1 + \hat{z}_2)/2, \qquad \hat{y}_2 = (\hat{z}_1 - \hat{z}_2)/2\kappa.$$

Now define

$$\bar{B} = (B_1 + B_2)/2, \qquad b = (B_1 - B_2)/2.$$

The map during the first half of the synchrotron oscillation is

$$\hat{y}_1(\theta) = e^{-i\bar{B}\theta}\left[\cos(b\theta)\hat{y}_{10} - i\kappa\sin(b\theta)\hat{y}_{20}\right] \tag{10}$$
$$\hat{y}_2(\theta) = e^{-i\bar{B}\theta}\left[\cos(b\theta)\hat{y}_{20} - \frac{i}{\kappa}\sin(b\theta)\hat{y}_{10}\right] \tag{11}$$

Apart from the overall phase evolution due to $\exp(-i\bar{B}\theta)$ and the small difference between b and b_0 these equations

* This work was supported by United States Department of Energy Contracts DE-SC0012704 and DE-AC02-76SF00515.
† blaskiewicz@bnl.gov

ISBN: 978-3-95450-168-7 doi:10.18429/JACoW-IPAC2015-MOPMN019

are the same as equations (5) and (6). It follows that the eigenvalues (λ) determining stability satisfy $Tr(M) = \lambda + 1/\lambda$ with

$$Tr(M) = 2\cos^2(b\pi/\nu_s) - \{\kappa^2 + 1/\kappa^2\}\sin^2(b\pi/\nu_s). \tag{12}$$

For $K > W$ equation (12) should give better answers than equation (7), since only normalization errors associated with the definitions of the eigenvectors are involved in equation (12). When $W > K$ and κ becomes imaginary things are not as clear. On the other hand note that

$$b - b_0 \approx (b_0/8)(K(K-W)/A^4) \sim b_0(\Delta\nu/\nu)^2,$$

so the effects are very small for any reasonable accelerator parameters. This small difference reinforces the utility of the single sideband approximation and allows us to use equation (7) with renewed confidence.

RESULTS

Equations (7) and (12) are amenable to significant analysis. We defer the details to a subsequent publication [3] and present some highlights. When $Tr(M_0) > -2$ the system is stable. Rearranging equation (7) yields the stability condition

$$\frac{\tan^2\left(\frac{\pi\sqrt{K(K-W)}}{2A\nu_s}\right)}{\frac{4K(K-W)}{W^2}} \equiv \frac{\tan^2(x\Upsilon/2)}{x^2} < 1, \tag{13}$$

where $\Upsilon = \pi W/2A\nu_s$, $x = 2\sqrt{K(K-W)}/W$. For $x = 0$ and small Υ, $\Upsilon = 2\pi\Delta\nu_W/\nu_s$ with $\Delta\nu_W$ the tune shift due to W. When $W > K$, x is imaginary and we use $\tan(iz) = i\tanh(z)$. Firstly note that there is no instability if $W = 0$. In this case the equations of motion do not change when particle 1 and 2 are interchanged and we have coupled oscillators with uniform focusing, like coupled pendulums. When $W > 0$ the equations of motion cannot be obtained from a Hamiltonian and instability is possible. Figure 1 shows a plot of equation (7) for fixed wake strength and synchrotron tune as a function of space charge tune shift. The unstable regions as a function of Υ and $\Delta\nu_{sc}/\nu_s$ are shown in Figure 2. Note that instability occurs for small Υ when $b_0\pi/\nu_s \approx (2n+1)\pi/2$. We have $Tr(M_0) = -2$ when $\Upsilon = 0$ and $b_0\pi/\nu_s = (2n+1)\pi/2$ and the unstable regions are continuously connected. It follows that all instabilities occur when $Tr(M_0) < -2$ and none occur with $Tr(M_0) > 2$.

While Υ and $\Delta\nu_{sc}/\nu_s$ are natural physical units the system stability is somewhat more clear in the units $U = \pi(K - W/2)/2A\nu_s$ and $V = \pi W/4A\nu_s$. In these units

$$Tr(M_0) = 2\cos^2\sqrt{U^2 - V^2} \\ -2(U^2+V^2)\frac{\sin^2\sqrt{U^2-V^2}}{U^2-V^2}. \tag{14}$$

Clearly, $Tr(M_0)$ is symmetric in both U and V and the power series expansion in U and V is well behaved over

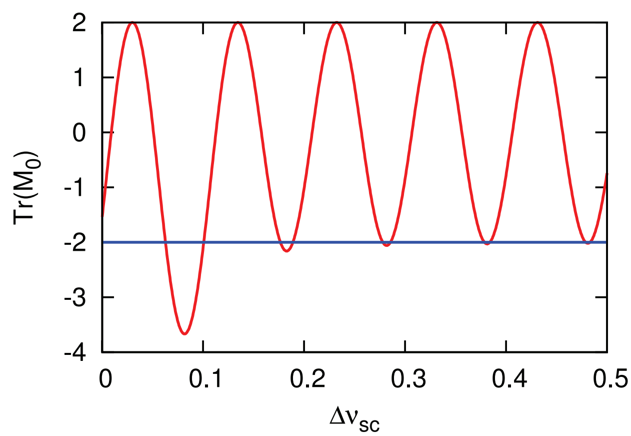

Figure 1: TrM_0 for $W/2A = 0.06$ and $\nu_s = 0.10$ as a function of $\Delta\nu_{sc} = K/2A$.

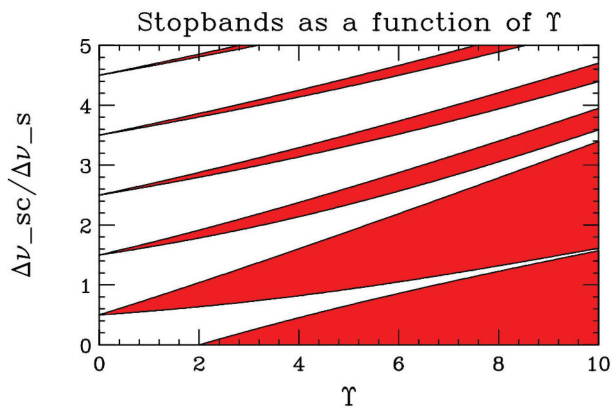

Figure 2: Unstable regions as a function of $\Upsilon = \pi W/2A\nu_s$ and $\Delta\nu_{sc}/\nu_s$.

the whole UV plane. Since $\sin^2(ix)/(ix)^2 = \sinh^2(x)/x^2$ and $|U^2 - V^2| \le U^2 + V^2|$ it is also clear that instability only occurs for $Tr(M_0) < -2$, verifying the argument in the previous paragraph. Figure 3 shows a contour plot of the growth rate in these variables. There are several instability regions with each region somewhat weaker than the one below it. It is interesting to note that the line $U = 0$ has $Tr(M_0) = 2$. For $V > U$ we have

$$Tr(M_0) = 2 - 4U^2\frac{\sinh^2\sqrt{V^2-U^2}}{V^2-U^2} \approx 2 - U^2e^{2V}/V^2,$$

where the approximations holds For $V \gg U$. The system goes from stable with $Tr(M_0) = 2$ to unstable with $Tr(M_0) < -2$ for very small values of U. The line $V = 0$ is stable owing to $W = 0$ so there is no wall impedance to drive instability.For $V \ll U$ we have

$$Tr(M_0) = (2 + 4V^2/U^2)\cos(2U) - 2V^2/U^2,$$

so when $\cos(2U) \approx -1$ only a small value of V is needed to make the system unstable.

The simplest continuum model describing this physics is an air bag distribution in a square well [4, 5]. While ana-

ISBN: 978-3-95450-168-7 doi:10.18429/JACoW-IPAC2015-MOPMN019

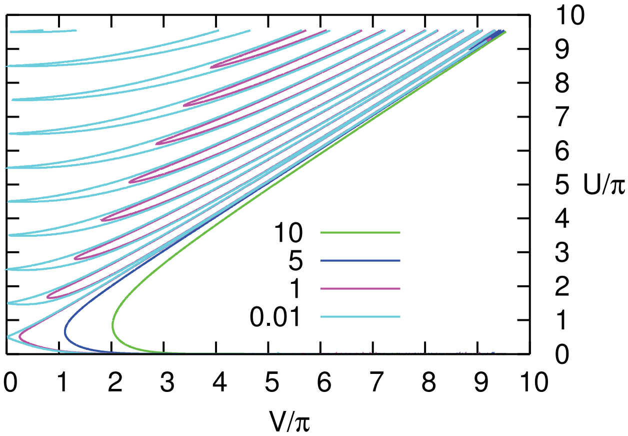

Figure 3: $\ln|\lambda|_{max}$ as a function of U and V.For $U = 0$, $Tr(M_0) = 2$ and the system is stable. For $V = 0$, $Tr(M_0) = 2\cos(2U)$ and the system is stable.

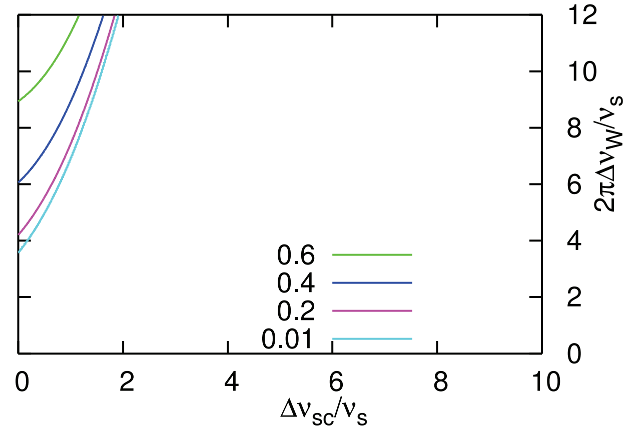

Figure 4: Growth rates for the air bag bunch in a square well as a function of $\Delta\nu_{sc}/\nu_s$ and $\Upsilon = 2\pi\Delta\nu_W/\nu_s$. All growth rates were $\leq 1 \times 10^{-8}$ outside the single bump with peak of order 1.

lytic results are not available it is possible to solve the relevant eigenvalue problem to machine precision on a computer. Figure 4 shows the growth rate as a function of $\Delta\nu_{sc}/\nu_s$ and Υ Note that the threshold value for instability with no space charge is $\Upsilon \approx 3.5$. For no space charge the two particle model has $\Upsilon = 2$ at threshold. Barring unstable regions of extent less than 0.02×0.02, the air bag model has only one unstable region in the parameter range plotted in Figure 3

REFERENCES

[1] M. Sands, SLAC-TN-69-8 (1969).

[2] A.W. Chao *Physics of Collective Beam Instabilities in High Energy Accelerators* Wiley, (1993).

[3] Y.H. Chin, A. Chao, M. Blaskiewicz *in preparation*.

[4] V.V. Danilov, E.A. Perevedentsev, 15th Iinternational Conference on High Energy Accelerators p1163 (1992).

[5] M. Blaskiewicz PRSTAB Vol. 1, 044201 (1998).