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STUDY OF EMITTANCE GROWTH CAUSED BY SPACE CHARGE AND LATTICE INDUCED RESONANCES

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Abstract

Emittance growth and beam loss in high intensity circular proton accelerators are one of the most serious issue which limit their performance. The emittance growth is caused by linear and nonlinear resonances of betatron/synchrotron oscillation due to lattice and space charge nonlinear force. The resonances are induced by errors in many cases. The space charge effects have been studied by computer simulations. Simulations with taking into account errors at random are consuming. We should first understand which resonances are serious. Resonance strength and resonance width induced by space charge and lattice nonlinearity is discussed with integrals along a ring like the radiation integrals. Emittance growth is evaluated by model with the resonance width to understand the mechanism.

INTRODUCTION

Particles move with experience of electro-magnetic field of lattice elements and space charge. We study slow emittance growth arising in a high intensity circular proton ring. We assume that the beam distribution is quasi-static, and each particle moves in the filed of the quasi-static distribution. Actually we concern about beam loss of 0.1-1% during a long term ($\approx$ 10,000 turns) in J-PARC MR. A halo is formed by the nonlinear force due to the electro-magnetic field. The halo, which consists of small part of whole beam, does not affect potential. Particle motion is described by a single particle Hamiltonian in the averaged field. This picture is not self-consistent for a distortion of beam distribution due to space charge force. Hamiltonian is separated by three parts for (1) linear betatron/synchrotron motion (μJ) (2) nonlinear component of the lattice magnets (U_{nl}) and (3) space charge potential (U).

$$H = \mu J + U_{nl} + U_{sc}. \tag{1}$$

where Hamiltonian is represented by action variables J and ϕ, which are Courant-Snyder invariant ($W = 2J$) and betatron phase, respectively.

Hamiltonian is expanded by Fourier series,

$$H = \mu J + U_{00(J)} + \sum_{m_x,m_y\neq 0} U_{m_x,m_y}(J)\exp(-im_x\phi_x - im_y\phi_y) \tag{2}$$

where Phase space structure near resonances are characterized by the resonance width. It is determined by their strength and tune slope for amplitude as follows [1],

$$\Delta J_x = 2\sqrt{\frac{U_{m_x,m_y}}{\Lambda}} \qquad \Lambda = \frac{\partial^2 U_{00}}{\partial J_x^2}. \tag{3}$$

EVALUATION OF RESONANCE WIDTH

Resonances Due to Space Charge Force

We first discuss the space charge potential U_{sc} [2]. Beam distribution is assumed to be Gaussian in transverse determined by emittance and Twiss parameters. U contains linear component, which gives a tune shift and Twiss parameter distortion. Twiss distortion is given by solving an envelope equation including linear space charge force self-consistently.

$$U_{sc} = \int ds' U_{sc}(s') = \frac{\lambda_p r_p}{\beta^2\gamma^3}\oint ds' \tag{4}$$

$$\int_0^\infty \frac{1 - \exp\left(-\frac{\beta_x(s')X(s,x')}{2\sigma_x^2+u} - \frac{\beta_y(s')Y(s,s')}{2\sigma_y^2+u}\right)}{\sqrt{2\sigma_x^2+u}\sqrt{2\sigma_x^2+u}}du$$

X and Y are normalized betatron coordinates at s' as

$$\begin{aligned} X(s,s') &= \sqrt{2J_x}\cos(\varphi_x(s') + \phi_x(s)) \\ Y(s,s') &= \sqrt{2J_y}\cos(\varphi_y(s') + \phi_y(s)). \end{aligned} \tag{5}$$

where $\varphi_{x,y}(s')$ is the betatron phase difference between s and s'.

The Fourier component, which correspond to resonance strength, is given by

$$\begin{aligned} U_{m_x,m_y}(J_x,J_y) = \frac{\lambda_p r_p}{\beta^2\gamma^3}\oint ds \int_0^\infty \frac{du}{\sqrt{2\sigma_x^2+u}\sqrt{2\sigma_y^2+u}} \\ \Big[\delta_{m_x 0}\delta_{m_y 0} - \exp(w_x - w_y)(-1)^{(m_x+x_y)/2} \\ I_{m_x/2}(w_x)I_{m_y/2}(w_y)e^{-im_x\varphi_x - im_y\varphi_y}\Big]. \end{aligned} \tag{6}$$

The tune slope $\partial^2 U_{00}/\partial J_x^2$ in Eq.(3) induced by space charge potential is obtained as follows. The tune slope is evaluated by $U_{00}(J_x,J_y)$ in Eq.(7).

$$\begin{aligned} U_{00}(J_x,J_y) = \frac{\lambda_p r_p}{\beta^2\gamma^3}\oint ds \int_0^\infty \frac{d\eta}{\sqrt{2+\eta}\sqrt{2r_{yx}+\eta}} \\ (1 - e^{-w_x-w_y}I_0(w_x)I_0(w_y)). \end{aligned} \tag{7}$$

where $r_{yx} = \sigma_y^2/\sigma_x^2$ and

$$w_x = \frac{\beta_x J_x/\sigma_x^2}{2+\eta}. \qquad w_y = \frac{\beta_y J_y/\sigma_y^2}{2+\eta/r_{yx}}. \tag{8}$$

$$\frac{\partial}{\partial J_x} = \frac{\beta_x/\sigma_x^2}{2+\eta}\frac{\partial}{\partial w_x}. \qquad \frac{\partial}{\partial J_y} = \frac{\beta_y/\sigma_x^2}{2r_{yx}+\eta}\frac{\partial}{\partial w_y}. \tag{9}$$

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The tune shift is given by derivative of U_{00} for J_{xy} as follows,

$$2\pi\Delta\nu_x = -\frac{\partial U_{00}}{\partial J_x} = -\frac{\lambda_p r_p}{\beta^2\gamma^3}\oint ds \frac{\beta_x}{\sigma_x^2}\int_0^\infty \frac{e^{-w_x-w_y}d\eta}{(2+\eta)^{3/2}(2r_{yx}+\eta)^{1/2}} \left[(I_0(w_x)-I_1(w_x))I_0(w_y)\right], \quad (10)$$

Similar formula is given for $2\pi\Delta\nu_y$.

The tune slope is given by second derivative of U_0 as follows,

$$\frac{\partial^2 U_{00}}{\partial J_x^2} = -2\pi\frac{\partial\nu_x}{\partial J_x} \quad (11)$$
$$= \frac{\lambda_p r_p}{\beta^2\gamma^3}\oint ds \frac{\beta_x^2}{\sigma_x^4}\int_0^\infty \frac{e^{-w_x-w_y}d\eta}{(2+\eta)^{5/2}(2r_{yx}+\eta)^{1/2}} \left[\left\{\frac{3}{2}I_0(w_x)-2I_1(w_x)+\frac{1}{2}I_2(w_x)\right\}I_0(w_y)\right],$$

where $I_0(x)' = I_1(x)$, $I_0(x)'' = (I_0(x)+I_2(x))/2$ are used. Similar formulae are given for $\partial^2 U_0/\partial J_x/\partial J_y$ and $\partial^2 U_0/\partial J_y^2$.

Figure 1 shows tune spread ($\Delta\nu_{x,y}(J_x, J_y)$), slope ($\partial^2 U_0/\partial J_x^2$), 4-th order resonance strength ($U_{4,0}$) and its width due to space charge force for J-PARC MR. The resonance width is visible size, 0.2ε, when $J_R = \varepsilon$.

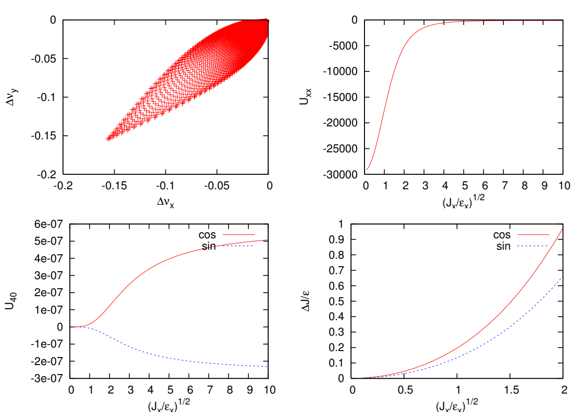

Figure 1: Tune spread ($\Delta\nu_{x,y}(J_x, J_y)$), slope ($\partial^2 U_0/\partial x^2$), 4-th order resonance strength ($U_{4,0}$) and its width due to space charge force as function of J_R.

Resonances Due to Nonlinear Magnets

Resonances and tune spread/slope are also induced by nonlinear magnets. One turn map is expanded by 12-th order polynomials for J-PARC MR. Taking at phase independent term, U_{00} is obtained as

$$\begin{aligned} U_{00}(\boldsymbol{J}) &= 3.43103\times 10^{14}J_x^6 + 7.36914\times 10^{14}J_x^5J_y + 7.17029\times 10^{11}J_x^5 + 2.34124\times 10^{15}J_x^4J_y^2 \\ &+ 1.70991\times 10^{12}J_x^4J_y + 1.43961\times 10^{8}J_x^4 + 4.48931\times 10^{15}J_x^3J_y^3 + 2.20917\times 10^{12}J_x^3J_y^2 \\ &+ 2.50211\times 10^{8}J_x^3J_y + 613899.J_x^3 + 3.33998\times 10^{15}J_x^2J_y^4 + 1.79716\times 10^{12}J_x^2J_y^3 \\ &+ 7.07531\times 10^{8}J_x^2J_y^2 + 809323.J_x^2J_y + 1095.71J_x^2 + 7.58773\times 10^{14}J_xJ_y^5 \\ &+ 5.7438\times 10^{11}J_xJ_y^4 + 4.55828\times 10^{8}J_xJ_y^3 + 650655.J_xJ_y^2 + 2096.06J_xJ_y \\ &+ 4.11283\times 10^{13}J_y^6 + 4.00294\times 10^{10}J_y^5 + 5.3027\times 10^{7}J_y^4 + 79924.4J_y^3 + 1106.98J_y^2 \end{aligned} \quad . \quad (12)$$

mx	my	Jx	Jy	|Um| (B0)	|Um| (B)	|Um| (BR)
1	0	3.6E-05	0.0E+00	4.84E-08	1.88E-07	1.86E-07
2	0	3.6E-05	0.0E+00	2.47E-08	4.55E-08	4.66E-08
1	1	1.8E-05	1.8E-05	1.28E-25	1.67E-26	4.01E-09
0	2	0.0E+00	3.6E-05	5.55E-09	3.91E-09	2.69E-09
3	0	3.6E-05	0.0E+00	5.46E-08	1.29E-07	1.32E-07
2	1	1.8E-05	1.8E-05	2.09E-25	1.42E-26	1.42E-07
2	-1	1.8E-05	1.8E-05	2.16E-25	4.52E-27	7.96E-08
1	2	1.8E-05	1.8E-05	4.66E-08	1.78E-07	1.83E-07
1	-2	1.8E-05	1.8E-05	1.48E-07	2.72E-07	2.72E-07
0	3	0.0E+00	3.6E-05	1.42E-25	1.59E-26	1.10E-07
4	0	3.6E-05	0.0E+00	2.50E-07	2.51E-07	2.51E-07
3	1	1.8E-05	1.8E-05	1.93E-26	2.52E-27	6.80E-09
3	-1	1.8E-05	1.8E-05	1.61E-26	4.97E-27	7.04E-10
2	2	1.8E-05	1.8E-05	2.49E-08	5.90E-09	5.58E-09
2	-2	1.8E-05	1.8E-05	1.27E-08	8.40E-09	8.03E-09
1	3	1.8E-05	1.8E-05	2.52E-26	5.66E-27	3.56E-09
1	-3	1.8E-05	1.8E-05	1.63E-26	1.10E-26	8.42E-10
0	4	0.0E+00	3.6E-05	1.20E-08	1.45E-08	1.42E-08

Table 1: $U_{m_x,m_y}(J)$ for lattice nonlinearity. U's are evaluated at J 3rd and 4-th column. The suffix, B0,B and BR means lattices without errors, lattice with measured beta and measured beta and coupling [3].

Figure 2 shows the tune shift and slope. Typical tune slope is $\partial^2 U_{00}/\partial x^2 = 1000 \sim 3000$. This value is similar for $U_{sc,00}$ at $J_x = 3^2\varepsilon$, namely tune slope of space charge is dominant for that of lattice nonlinearity at $J < 9\varepsilon(3\sigma)$, vice versa.

Resonance strength due to lattice nonlinearity is obtained by the one turn map. Table 1 shows the resonance strength $U_{m_x,m_y}(J)$. up to 4-th.

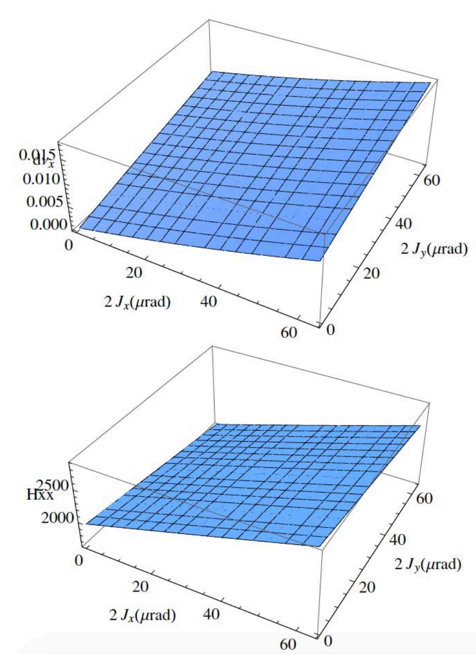

Figure 2: Tune spread, $\partial^2 U_{00}/\partial x^2$ induced by lattice nonlinearity.

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TOY MODEL WITH THE TUNE SLOPE AND RESONANCE STRENGTH

We study emittance growth for an accelerator model with a given tune slope and resonance strength. This is an example of Hamiltonian,

$$H = \mu_0 J + \left(J + \frac{e^{-2aJ}}{2a}\right) + bJ\cos m\phi. \qquad (13)$$

The tune shift is given by

$$\mu = \frac{\partial H}{\partial J} = \mu_0 + (1 - e^{-2aJ}). \qquad (14)$$

For small amplitude tune shift $2aJ$, where $a > 0$. The tune slope is given by

$$\frac{\partial^2 H}{\partial J^2} = 2ae^{-2aJ}. \qquad (15)$$

Half width of the resonance is expressed by

$$\Delta J = \sqrt{\frac{2bJ_R}{ae^{-2aJ_R}}}. \qquad (16)$$

Symplectic integration is performed by $H(J, \phi)$ as follows,

$$\begin{aligned} J_{n+1} &= \frac{J_n}{1 - bm\sin m\phi_n} \qquad (17) \\ \phi_{n+1} &= \phi_n + \mu + (e^{-2aJ_{n+1}} - 1) + b\cos m\phi_n, \end{aligned}$$

where J_n and ϕ_n are those of n-th turn.

We study two cases of parameters,

- $a = 0.5$, $b = 0.002$, $m = 4$, $\mu = 2\pi \times 0.203$
- $a = 0.5$, $b = 0.0002$, $m = 4$, $\mu = 2\pi \times 0.203$

The resonance widths are given as (1) $\Delta J = 0.07$ and (2) =0.02. The betatron amplitude, where the resonance hits, is $J_R = 0.38$.

The model is tracked using the two sets of parameters. Figure 3 shows phase space trajectories. 4-the order resonance is seen, and their position (J_R) and widths agree with the formula, Eqs.(14) and (16)

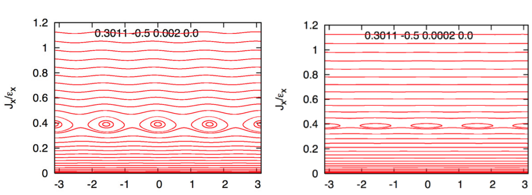

Figure 3: Phase space trajectory for the model map, Eq.(17). Left and right plots correspond to parameters (1) and (2), respectively.

Tune spread area modulates due to synchrotron oscillation. To study the effect, the strength of tune shift term a is made a modulation as

$$a = \frac{a_0}{2}(1 + \cos 2\pi\nu_s n). \qquad (18)$$

The resonant amplitude move to larger amplitude for small a. The model does not match to space charge force in this point. This model should be improved in the future. Figure 4 shows phase space plot taking into account of the effective synchrotron motion. Chaotic area drastically increases due to the synchrotron motion. Figure 5 shows the emittance growth of the model with Eqs.(17) and (18). We can see the emittance growth depending on the resonance width.

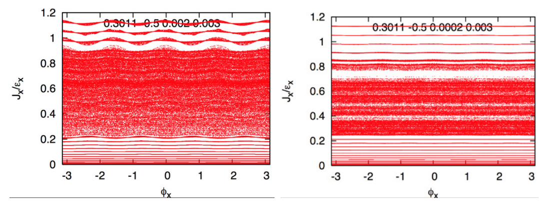

Figure 4: Phase space trajectory for the model map taking into account of effective synchrotron motion, Eqs.(17) and (18). Left and right plots correspond to parameters (1) and (2), respectively.

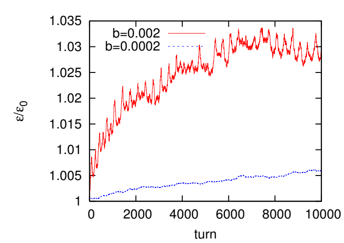

Figure 5: Emittance growth of the model with Eqs.(17) and (18).

SUMMARY

Tune slope and resonance strength induced by lattice and space charge nonlinear force were evaluated by integrals along ring. The resonance width which characterize emittance growth is estimated by them. A simple model with the resonance information is examined to study emittance growth. Synchrotron motion is taken into account of changing space charge tune shift for z. Enhancement of the emittance growth was evaluated.

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