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COHERENT SYNCHROTRON RADIATION FIELD AND THE ENERGY LOSS IN A WAVY BEAM*

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Abstract

The synchrotron radiation will be coherent when the wavelength of the radiation can be compared with the bunch length. There are two approaches to produce Coherent Synchrotron Radiation (CSR) on a storage ring. One is to compress the bunch length, the other one is to produce a wavy beam which has high spatial repetition along the longitudinal direction. The latter one can expand the radiation frequency range of a light source. However, CSR can bring nonlinear effect which brings in extra instability. The Liénard-Wiechert potentials in three-dimensional space may have very complicated forms. The most common way to investigate CSR is numerical method. This paper try to use a simple model to obtain energy loss of the electrons in theory.

PHYSICAL PICTURE

Assuming an electron moves along a fixed circular orbit of radius ρ with a constant speed $|\vec{\beta}| = \beta$. At the present moment, the electron locates at point P. We want to know the fields around point P. Radiation field in the orbit plane is discussed in [1]. To simplify the question, we assume the observation point A just above or below the trajectory. Thus, we get a two dimensional model. The field of point A is emitted at an earlier time when the electron located at point P'. The relations between P,P' and A are as shown in Fig. 1.

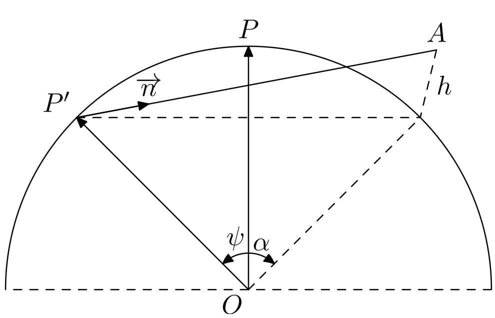

Figure 1: Diagram of the 2D model. ψ is the retarded angle, α is the azimuthal angle between P and P', h is the height of A relative to the orbit plane. Here $h > 0$ means the observation point A is above the orbit plane and $\alpha > 0$ means A is ahead of the electron present position P. ψ is always positive and $-\pi < \alpha < \pi$.

According to the geometric relationship, we can get the retarded equation:

$$\frac{\psi^2}{\beta^2} = (\frac{h}{\rho})^2 + 4\sin^2(\frac{\psi + \alpha}{2}). \qquad (1)$$

The retarded equation is nonlinear, so it is generally not possible to obtain an exact answer. However, under some approximations, we can get some meaningful analytic solutions.

SOLUTION OF THE RETARDED EQUATION

Equation (1) shows that the retarded angle ψ depends on the electron's energy γ, the longitudinal azimuthal angle α and the height between the observation point to the orbit plane. In other words: from equation (1), we can get the numerical solution as shown in Fig. 2 and Fig. 3.

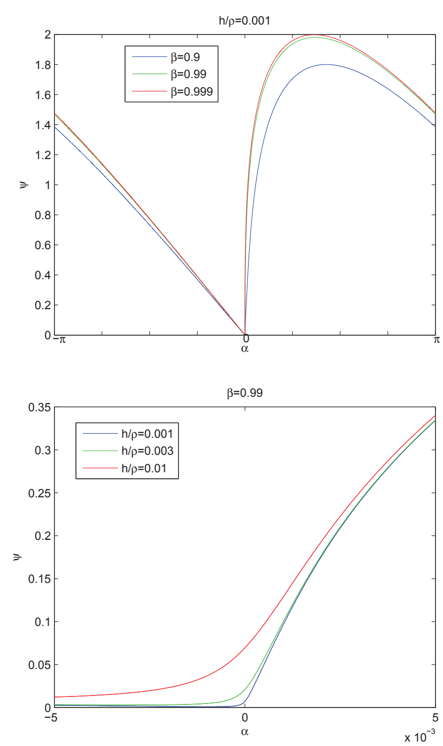

Figure 2: Retarded angle ψ as a function of α. The upper shows how ψ varies with the whole α when $h/\rho = 0.001$ for different β(or γ). And the lower shows small α when $\beta = 0.99$ for different h/ρ.

The conclusions are: (a)$\psi = \alpha_0$ when $\alpha = -\alpha_0$, here $\alpha_0 = h\beta/\rho$; (b)ψ is bounded; (c)ψ grows rapidly when $\alpha > 0$; (d)ψ is weakly related to γ when γ is large enough; (e)$(\psi + \alpha)/2 < 0$ when $\alpha < 0$ and vice verse.

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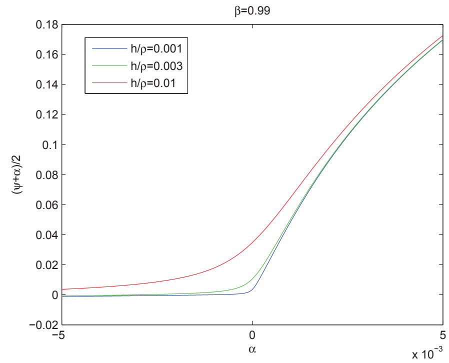

Figure 3: $(\psi+\alpha)/2$ as a function of small α when $\beta = 0.99$ for different h/ρ.

Back to the question raised at first, what we really care about is the field at the observation point close to the electron position. So α and α_0 are small. However, ψ doesn't have such restrictions at all.

when $\psi \ll 1$.

The following discussion bases on the ultrarelativistic approximation $\gamma \gg 1$. $a \sim o(b)$ means that a and b are small and $|a/b| \approx 0$, while $a \sim b$ means that $|a/b| < \infty$ and $|b/a| < \infty$. Introduce $x = \psi + \alpha$, $x \ll 1$. Then we use *padé* approximation to replace $\cos x$. Substituting it into Eq. (1):

$$\begin{aligned} 0 =& \frac{1}{12}(\gamma x)^4 + \left[1 + \frac{(\gamma\alpha)^2 - (\gamma\alpha_0)^2}{12}\right](\gamma x)^2 \\ &- 2(\gamma^3\alpha)(\gamma x) + (\gamma^2\alpha)^2 - (\gamma^2\alpha_0)^2. \end{aligned} \tag{2}$$

To obtain this result, $x^2 \ll 1$ has been used.

In the above equation, if we choose $\tilde{\alpha}_0 = \gamma^2\alpha_0 \sim 1$ and $\tilde{\alpha} = \gamma^3\alpha \sim 1$, then: $\tilde{x} = \gamma x \sim 1$. Conversely, if $x \sim \frac{1}{\gamma}$ or higher order, $\alpha_0 \sim o(\frac{1}{\gamma})$ and $\alpha \sim o(\frac{1}{\gamma^2})$.

We can prove that by making $\alpha = 0$ or $\alpha_0 = 0$. Substituting $\alpha = 0$ into Eq. (2), then we can get an analytical solution:

$$x_1 = \sqrt{\frac{12\alpha_0^2}{(\frac{6}{\gamma^2} - \frac{\alpha_0^2}{2}) + \sqrt{(\frac{6}{\gamma^2} - \frac{\alpha_0^2}{2})^2 + 12\alpha_0^2}}} \tag{3}$$

if $\alpha_0 \sim \frac{1}{\gamma}$, $x_1 \sim 1$ which violates the previous assumption. Especially when $\alpha_0 \sim o(\frac{1}{\gamma^2})$, $x_1 = \gamma\alpha_0$.

Substituting $\alpha = 0$ into Eq. (2) with the same procedure, the model becomes 1D. And this situation had been discussed in [2]:

$$x_2 = \begin{cases} \frac{2}{\gamma}(\Omega^{1/3} - \Omega^{-1/3}) & \alpha > 0 \\ \frac{\alpha}{2} & \alpha < 0 \end{cases} \tag{4}$$

where:

$$\Omega = \frac{\gamma^3\alpha}{2} + \sqrt{1 + (\frac{\gamma^3\alpha}{2})^2}. \tag{5}$$

if $\gamma^3\alpha \gg 1$, $x_2 \sim \alpha^{\frac{1}{3}}$. For an ultrarelativistic electron, x_2 is larger than $\frac{1}{\gamma}$.

In the previous case, we can drop α and α_0 in Eq. 2, then we find the solution $x = x_2$. This result means the field is the same with the point on the trajectory when the observation point is close enough to the orbit plane.

When $\alpha < 0$, from Fig 3 we know that x can be very small. So we drop the x^4 term in Eq. (2) around $\alpha = -|\alpha_0|$. Then x can be solved:

$$x_3 = \gamma^2\alpha + \gamma\sqrt{\gamma^2\beta^2\alpha^2 + \alpha_0^2} \tag{6}$$

Here $\alpha \sim \alpha_0 \sim \frac{1}{\gamma^2}$ has been used. $x_3|_{\alpha_0=0} \approx \frac{\alpha}{2}$ which is consistent with 1D case and $x_3|_{\alpha=0} = \gamma\alpha_0$ is consistent with x_1.

SINGLE PARTICLE WAKE FIELD

From the Liénard-Wiechert potentials, the electric field [3] from a moving electron is

$$\vec{E} = e\left[\frac{\vec{n} - \vec{\beta'}}{\gamma^2(1 - \vec{n}\cdot\vec{\beta'})^3R^2}\right] + \frac{e}{c}\left[\frac{\vec{n} \times \{(\vec{n} - \vec{\beta'}) \times \dot{\vec{\beta'}}\}}{(1 - \vec{n}\cdot\vec{\beta'})^3R}\right], \tag{7}$$

where R is the distance between the observation point and the radiation emitted location. The tangential component of the electric field is:

$$E_s = \frac{ke\beta^3}{\rho^2}\left[\frac{(u-1)}{\gamma^2\psi^2} + v - u + \beta^2 uv\right], \tag{8}$$

here

$$u = \frac{\sin(\psi+\alpha)}{\psi}, v = \frac{1 - \cos(\psi+\alpha)}{\psi^2}, k = \frac{1}{(1-\beta^2 u)^3}. \tag{9}$$

when $\alpha = 0$.

At $\alpha = 0$, Eq. (8) doesn't contain any singular term. Substituting expansions of u and v into Eq. (8)

$$E_s^{rad} = \frac{e\gamma^4\beta^3}{\rho^2}(-\frac{2}{3} + \frac{1}{3}\gamma^2\psi^2) \tag{10}$$

when $\alpha_0 = 0$.

This case is discussed in [4] in detail. The forward wake field is much bigger than the backward wake field.

$$E_s^{rad} = \begin{cases} -\frac{4}{3}\frac{\gamma^4 e}{\rho^2} + \frac{14}{3}\frac{e}{\rho^2}\gamma^6 x^2 & \alpha > 0, \alpha \ll \frac{1}{\gamma^3}. \\ \frac{e}{8\rho^2} + \frac{e}{16\rho^2}x^2 & \alpha < 0 \end{cases} \tag{11}$$

This is consistent with the instantaneous power loss of the emitting electron from the Larmor formula in the point charge model [2].

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when $\alpha > 0$, $\alpha \sim \frac{1}{\gamma^3}$ *or* $\alpha_0 \sim \frac{1}{\gamma^2}$.

In this situation, $\frac{\alpha}{\psi} \sim \frac{1}{\gamma^2}$. Expand u and v at ψ, and Substitute it into Eq. (8):

$$\begin{aligned} E_s^{rad} &= \frac{e\beta^3}{\rho^2}\frac{-\frac{2}{3}\gamma^4 + \gamma^2(\frac{2\tilde{\psi}^2}{15} + \frac{\tilde{\psi}^4}{180})}{(1+\frac{1}{6}\beta^2\tilde{\psi}^2)^3} + \frac{e\beta^3}{\rho^2}\frac{\tilde{\alpha}}{\gamma^2\tilde{\psi}^3} \\ &\approx -\frac{2\gamma^4}{3}\frac{e\beta^3}{\rho^2}\frac{1}{(1+\frac{1}{6}\beta^2\tilde{\psi}^2)^3}. \end{aligned} \tag{12}$$

when $\alpha > 0, \alpha_0 \ll \frac{1}{\gamma^2}$ *and* $\alpha \ll \frac{1}{\gamma^3}$.

The first term in Eq. (8) is singular. It is hard to remove the singularity. Since the observation is quite close to the electron, and the radiation field is continuous besides the present position of the electron. A reasonable assumption is $E_s^{rad} = -\frac{4}{3}\frac{e\gamma^4\beta^3}{\rho^2}$.

In remote region, the electric field [4] is expressed as

$$E_s^{rad} = \frac{2e}{(3\alpha)^{\frac{4}{3}}\rho^2}. \tag{13}$$

WAKE FIELD OF A WAVY BEAM

Considering a relativistic electron beam which has a shape of a wiggle in the vertical direction. This beam has a very high spatial repetition frequency along the closed orbit. The far-field radiation of such beam is discussed in [5]

The beam distribution in longitudinal-vertical plane can be described as

$$\vec{r}(t,\tau) = \vec{r_0}(t-\tau) + y_0 \sin\omega_y\tau\vec{e}_y, \tag{14}$$

where $\vec{r_0}(t)$ is the closed orbit of the reference electron which usually is a circle. ω_y is the spatial repetition frequency, τ is the longitudinal time displacement. If the frequency ω_y is really high so that $\frac{2\pi c}{\omega_y\rho} \ll \frac{1}{\gamma^3}$, the electrons in a wavelength will loss the same energy. So CSR doesn't affect the bunch emittance. However, for an ultra-relativistic beam, it is very difficult to produce such structure. In the following discussion, we suppose $\frac{y_0}{\rho} \sim \frac{1}{\gamma}$, $\omega_y \sim \frac{c\gamma^2}{\rho}$.

We already have the radiation field,

$$\begin{aligned} E_0 &= -\frac{4}{3}\frac{\gamma^4 e}{\rho^2}, \\ E_1 &= \frac{e\beta^3}{\rho^2}\frac{-\frac{2}{3}\gamma^4}{(1+\frac{1}{6}\beta^2\tilde{\psi}^2)^3}, \\ E_2 &= \frac{2e}{(3\alpha)^{\frac{4}{3}}\rho^2}. \end{aligned} \tag{15}$$

The distribution of the density is $\rho(\tau)$. System of N identical equidistant charges q moving with constant velocity v along an arbitrary closed path does not radiate in the limit of $N \to \infty$ and $Nq = const$, and the electric and magnetic fields of the system are the usual static values [4]. If the electron beam is in steady state, the radiation field satisfies $E_s(\tau) = E_s(\tau + \frac{2\pi}{\omega_y})$. So

$$\int_0^{2\pi/\omega_y} E_s(\tau)d\tau = 0. \tag{16}$$

The longitudinal period of the beam is $T = 2\pi/\omega_y$, then

$$\begin{aligned} E_s^{rad} &= \int_{\tau_0-T}^{\tau_2} E_2\rho(\tau)d\tau \\ &+ \int_{\tau_2}^{\tau_1} E_1\rho(\tau)d\tau + \int_{\tau_1}^{\tau_0} E_0\rho(\tau)d\tau \end{aligned} \tag{17}$$

Because $\tau_1 - \tau_2 \gg (\tau_0 - \tau_1)$ and E_0 has the same order with E_1. Then,

$$E_s^{rad} = \int_{\tau_0-T}^{\tau_2} E_2\rho(\tau)d\tau + \int_{\tau_2}^{\tau_0} E_1\rho(\tau)d\tau \tag{18}$$

where τ_2 satisfies

$$\int_{\tau_0-T}^{\tau_2} E_2 d\tau + \int_{\tau_2}^{\tau_0} E_1 d\tau = 0. \tag{19}$$

Eqs. (15)(18)(19) together with the retarded Eq. (2) can be used to calculate the CSR field. Here gives a simple result in Fig. 4.

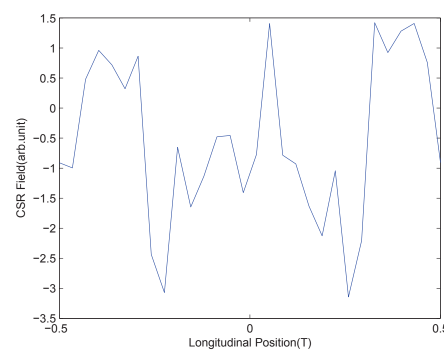

Figure 4: CSR field varies with the longitudinal position of a wavy beam in one period. The longitudinal period $T = \frac{2\pi}{\omega_y}$ has the same order with $\frac{c}{\gamma^2\rho}$. The vertical amplitude has the same order with $\frac{\rho}{\gamma}$

.

It shows the energy loss is nonlinear Correlated with the longitudinal position. Because of the periodicity, the CSR field in a wider range need more calculation.

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