

# PHASE SPACE DENSITY AS A MEASURE OF COOLING PERFORMANCE FOR THE INTERNATIONAL MUON IONIZATION COOLING EXPERIMENT

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## Abstract

The International Muon Ionization Cooling Experiment (MICE) [1] is an experiment to demonstrate ionization cooling of a muon beam in a beamline that shares characteristics with one that might be used for a muon collider or neutrino factory. I describe a way to quantify cooling performance by examining the phase space density of muons, and determining how much that density increases. This contrasts with the more common methods that rely on the covariance matrix and compute emittances from that. I discuss why a direct measure of phase space density might be preferable to a covariance matrix method. I apply this technique to an early proposal for the MICE final step beamline. I discuss how matching impacts the measured performance.

## INTRODUCTION

The analysis of the MICE particle trajectories must provide a numerical measure of the cooling performance, and must ensure that this performance measure is unbiased. To provide an example of a potential difficulty, I first examine two possible measures of cooling performance; the first is an increase in the phase space density

$$N_f \epsilon_{6i} / (N_i \epsilon_{6f}) \quad (1)$$

where the  $i$  subscripts refer to the initial distribution, the  $f$  subscripts to the final distribution,  $N$  refers to the number of particles, and  $\epsilon_6$  refers to the 6-D phase space emittance, defined to be the square root of the determinant of the 6-D second moment matrix. A different measure could be the luminosity increase:

$$N_f^2 \sqrt{\epsilon_{6i}} / (N_i^2 \sqrt{\epsilon_{6f}}) \quad (2)$$

This is generally a stronger condition than the increase in phase space density, but may be a more appropriate measure for a muon collider as it measures the eventual increase in luminosity one might expect (though there are arguments for a different contribution from the longitudinal emittance).

I use an early version [2] of the MICE final step lattice. The design parameters are given in Tables 1–3. There are two 201.25 MHz cavities with a gradient of 16 MV/m, a length of 434.62 mm, centered at  $\pm 281.81$  mm. The beam distribution will always be launched from  $-4050$  mm and

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Table 1: Solenoid coil geometry and currents [2] used in my simulations. Only the upstream half of the coils are listed; the downstream half is reflected about longitudinal position 0 and has opposite currents.

Long. Center (mm)	Length (mm)	Inner Radius (mm)	Outer Radius (mm)	Current (A/mm <sup>2</sup> )
-4800.00	110.60	258.0	325.8	135.18
-4050.00	1314.30	258.0	280.1	152.44
-3300.00	110.60	258.0	318.9	127.37
-2900.00	199.50	258.0	288.9	113.12
-2461.00	201.30	258.0	304.2	123.04
-1202.75	213.30	267.0	361.8	40.00
-797.25	213.30	267.0	361.8	40.00

Table 2: Apertures [2] used in my simulation. Only the apertures upstream of the center are listed; the downstream apertures are identical.

Center (mm)	Length (mm)	Radius (mm)
-3592	2664	200
-1000	844	210
0	65	600

analyzed at  $+4050$  mm. The beam has a total momentum of 200 MeV/c, and I only consider transverse dimensions. Simulations are performed using ICOOL [3].

To illustrate the potential difficulties I perform a simple simulation: I launch a beam with a Gaussian distribution in transverse phase space, matched to the 4.14565 T solenoid field at the launch point, including its angular momentum, with a given normalized emittance. I then compute the normalized emittance at the end using

$$\sqrt{\sigma_{xx}\sigma_{p_x p_x}\sigma_{yy}\sigma_{p_y p_y}} - \sigma_{x p_x}\sigma_{y p_y} - L^2/4 \quad (3)$$

$$L = \sigma_{x p_y} - \sigma_{y p_x} \quad (4)$$

where  $\sigma_{ij}$  are the corresponding elements of the second order moment matrix. I then use Eq. (1) to compute the increase in phase space density. Figures 1 and 2 show the results. Whether one sees cooling or not depends on the choice of the initial emittance, and there is no clear reason to choose one emittance over another. For small emittances,

Table 3: Materials intercepting the beam [2] in my simulation. Only the materials upstream of the center are listed; the downstream materials are identical. The lithium in LiH is almost exclusively Li6.

Material	Center (mm)	Length (mm)
Be	-499.31	0.38
Be	-64.31	0.38
LiH	0	65

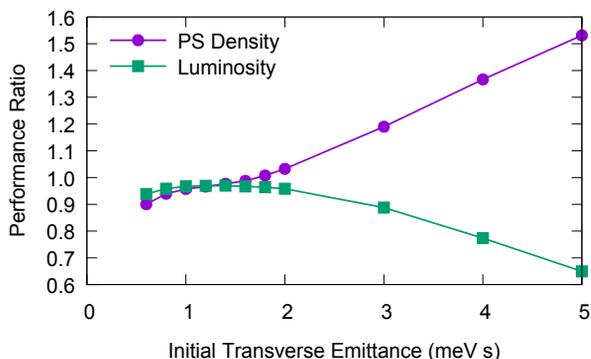


Figure 1: Ratio of final phase space density to initial phase space density [Eq. (1)], and the same for luminosity [Eq. (2)], as a function of the initial normalized transverse emittance.

multiple scattering in the absorber causes the emittance to grow more rapidly than the emittance is decreased by the average energy loss. For large emittances, the tails of the distribution are truncated on the apertures, giving the false appearance of a very large phase space density increase. If instead one uses Eq. (2) as a measure, one finds a clear optimum, but it does not show an increase in that measure (the experiment is not designed to increase this measure).

The weakness here is the use of the second order moment matrix as a measure of the phase space density, and the limited focusing and cooling potential of the lattice. I will

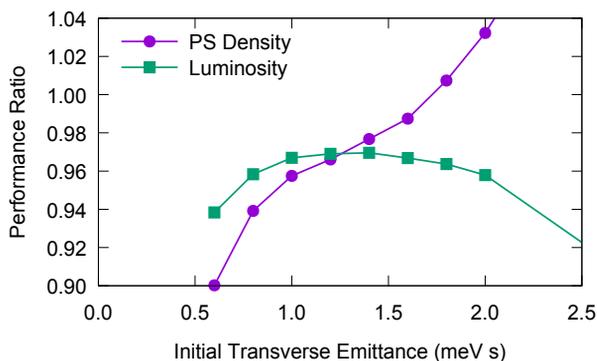


Figure 2: Detail of Fig. 1.

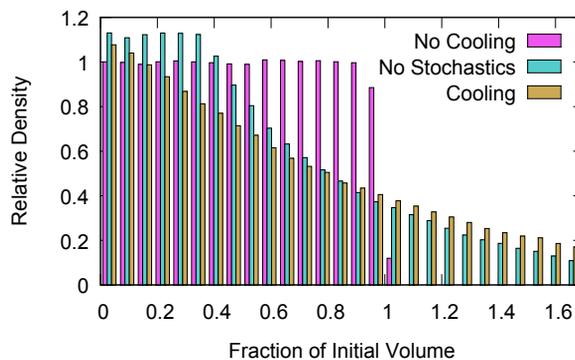


Figure 3: Particle count in histogram bins relative to what would be expected for a linear system with no cooling. Horizontal axis corresponds to  $(J_x + J_y)/J$ . 262144 particles were launched with  $J = 1.0$  meV s. Runs are with either no absorbers, only the average energy loss included, or a full simulation with energy loss, energy straggling, and multiple scattering included.

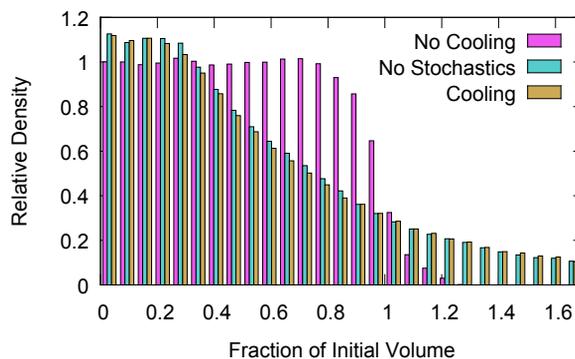


Figure 4: As for Fig. 3, but for  $J = 9.2$  meV s.

demonstrate that the lattice increases phase space density by directly looking at the phase space density itself.

## PHASE SPACE DENSITY

I launch an initial distribution with a uniform density in the 4-dimensional transverse phase space, then count particles at the end in bins which have a fixed volume in phase space. The number of particles in a bin, divided by the phase space volume of the bin, divided by the initial phase space density, gives the increase ratio in phase space density.

I have not worked out a number of details of the practical implementation of this. It is in principle possible to achieve a uniform phase space density to some accuracy by an initial blind particle selection; an algorithm must be worked out to do so. An uncertainty analysis is necessary to determine the statistical significance of the particle counts within the bins. This paper will only illustrate the concept of the technique.

I launch a distribution of particles uniformly distributed in transverse phase space by distributing particles uniformly in a triangle in action space defined by  $J_x \geq 0$ ,  $J_y \geq 0$ ,  $J_x + J_y < J$ , and angle space  $0 \leq \Phi_{x,y} < 2\pi$ . These are

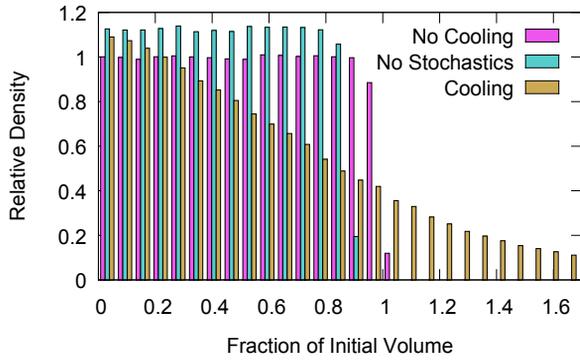


Figure 5: Figure 3, after optimizing the transformation to  $J_x + J_y$ .

converted into initial coordinates via

$$x = \sqrt{\frac{4J_x}{e|B_s|}} \cos \Phi_x \quad y = \sqrt{\frac{4J_y}{e|B_s|}} \cos \Phi_y \quad (5)$$

$$p_x = -\sqrt{e|B_s|J_x} \sin \Phi_x + \frac{eB_s}{2} y \quad (6)$$

$$p_y = -\sqrt{e|B_s|J_y} \sin \Phi_y - \frac{eB_s}{2} x \quad (7)$$

with  $B_s = 4.14565$  T. Note that the momenta here are kinetic momenta.

I compute  $J_x + J_y$  for each particle using

$$J_x + J_y = \frac{e|B_s|}{4} (x^2 + y^2) + \frac{1}{e|B_s|} \left[ \left( p_x - \frac{eB_s}{2} y \right)^2 + \left( p_y + \frac{eB_s}{2} x \right)^2 \right] \quad (8)$$

where in this case  $B_s$  is  $-4.14565$  T, the field at 4.05 m. The particles are then assigned to a histogram bin by computing

$$\left\lfloor n \left( \frac{J_x + J_y}{J} \right)^2 \right\rfloor \quad (9)$$

This assignment gives each bin the same phase space volume. If the system were perfectly linear with no cooling, only first  $n$  bins would be filled, and they would, on average, all contain the same number of particles. For the real system, more than  $n$  bins could be filled. If there are  $N$  total particles launched, a bin having more than  $N/n$  particles, by a statistically significant amount, indicates that phase space density was increased. The result of this analysis is shown in Figs. 3 and 4. Without absorbers, the final distribution is close to having all  $n$  bins filled with  $N/n$  particles, except near bin  $n$ , where nonlinearity causes deviation from that, more for a large emittance distribution than a small emittance distribution. When absorbers are included, one sees an excess of particles in the low amplitude bins, giving the expected cooling behavior. Furthermore, when stochastic effects are included, there is a gain in only the lowest amplitude bins, a reduction in higher amplitude bins, and a significant number

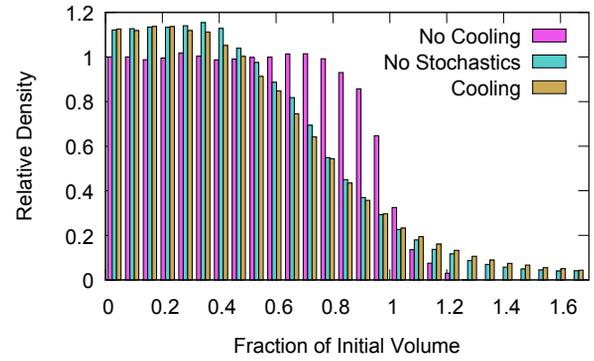


Figure 6: Figure 5, but for  $J = 9.2$  meV s.

of particles in bins for  $J_x + J_y > J$ . This effect is more pronounced when  $J$  is small. This is the expected effect from multiple scattering, since the amount of scattering is independent of the particle amplitude. What is unexpected, however, is that when stochastic effects are turned off but energy loss in absorbers is included, there are still particles for  $J_x + J_y$  well beyond  $J$ , and a corresponding reduction in the number of particles for  $J_x + J_y$  below  $J$ .

This latter effect arises from a betatron mismatch generated by the absorber. I address this by allowing a general cylindrically symmetric transformation to action variables:

$$J_x + J_y = \frac{e|B_{s1}|}{4} (x^2 + y^2) + \frac{1}{e|B_{s1}|} \left\{ \left[ p_x - \frac{e}{2} (B_{s0}y + |B_{s1}|\alpha x) \right]^2 + \left[ p_y + \frac{e}{2} (B_{s0}x - |B_{s1}|\alpha y) \right]^2 \right\} \quad (10)$$

where  $B_{s0}$ ,  $B_{s1}$ , and  $\alpha$  are chosen to maximize the number of particles with  $J_x + J_y < J$ . The results are shown in Figs. 5 and 6. The results without stochastic effects are as expected for the  $J = 1.0$  meV s distribution, and are improved but not completely satisfactory for the  $J = 9.2$  meV s distribution. The results including stochastic effects are also improved, in that for the same set of particles, there are more histogram bins showing a phase space density increase. Choosing a better criterion for matching may improve the results for large emittance distributions; as evidence of this, the results appeared to be better when I fixed  $\alpha = 0$ .

## REFERENCES

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