EXACT SOLUTIONS OF THE VLASOV EQUATION IN MAGNETIC FIELD *

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Abstract

Stationary self-consistent distributions for charged particle beam in magnetic field are considered. These distributions can be regarded as formal solutions of the Vlasov equation which is formulated in the covariant form.

COVARIANT FORM OF THE VLASOV EQUATION

Dynamics of a charged particle beam can be described as dynamics of all individual particles composing it. Another model of the beam dynamics is the Vlasov equation which is an integro-differential equation for density of the particle distribution in the phase space.

Let us formulate the Vlasov equation in covariant form following the works [1, 2]. Firstly, let us specify a reference frame and introduce the phase space $\mathcal{M}$ as the tangent bundle of the configuration space associated with given reference frame [3, 4]. If particles always lie on the same surface S in the phase space, or distribution density does not depend on some coordinate, then the phase space $\mathcal{M}$ can be taken as coresponding subspace of the initial phase space.

Define the particle distribution density in the phase space (phase density) [1, 2] as such differential form $n(t,q)$ of degree p that for any open subdomain $G \subset \mathcal{M}$

$$\int_D n = N_G. \tag{1}$$

Here N_G is a number of particles in G, t is the time, $q \in \mathcal{M}$. If particle lie in some open subdomain of $\mathcal{M}$, then $p = \dim \mathcal{M}$, $D = G$. If all particles lie on some surface S in the phase space, then $p = \dim S, D = S \cup D$. The latter case includes as a particular case a set of point-like particles. In this case, the phase density is a form of degree 0, i.e. a scalar function, and integration over D is the summation over all particles lying in G.

In all cases the Vlasov equation can be written in the form [1, 2]

$$n(t + \delta t, F_{f,\,\delta t} q) = F_{f,\,\delta t} n(t,q). \tag{2}$$

Here $F_{f,\,\delta t} q$ denotes Lie dragging along vector field f [4], which is defined by the dynamics equation $dq/dt = f$, and depends on an external field and the self field.

Consider the case when particle distribution is described by the form of maximal degree. Assume that its single component $\tilde{n}$ is continuously differentiable on phase coordinates. Ho do this component change at some point q of the phase space depending on the time t?

Let the phase density at some instance t at a point q be equal to $n(t,q)$. At the instance $t + \delta t$ in this point, it will be equal to $n(t+\delta t, q) = F_{f\,\delta t} n(t, F_{f\,-\delta t} q)$, as the phase density change according to equation (2). Introduce the derivative of a differential form on the parameter t as a form which components are derivatives of corresponding components on t. Then we obtain the Vlasov equation in the form

$$\frac{\partial n}{\partial t} = \lim_{\delta t \to 0} \frac{n(t+\delta t, q) - n(t,q)}{\delta t} = -\mathcal{L}_f n(t,q). \tag{3}$$

Here $\mathcal{L}_f n(t,q)$ denotes the Lie derivative of the phase density along the vector field f. Components of the Lie derivative of a differential form of degree p can be written in the form

$$(\mathcal{L}_f T)_{i_1 \dots i_p} = \frac{\partial T_{i_1 \dots i_p}}{\partial q^k} f^k + \frac{\partial f^j}{\partial q^{i_1}} \cdot T_{j\, i_2 \dots i_p} + \tag{4}$$

$$\dots \frac{\partial f^j}{\partial q^{i_p}} \cdot T_{i_1 \dots i_{p-1}\, j} \tag{5}$$

(summation is meant on coincident indices).

LONGWISE UNIFORM BEAM

Consider stationary longitudinally uniform axially symmetric beam in a uniform longiduninal magnetic field [7-13]. Uniformity means that phase distribution does not depend of longituninal coordinate z. Assume also that longitudinal velocities v^z of all particles are the same. Then we can consider particle distribution in 4-dimensional phase space. Axial symmetry means that the phase density and density in the configuration space do not depend of azimuthal angle φ. Then electric field potential U depends only on radial coordinate $r : U = U(r)$. In this case, particle dynamics equations give the first integrals of motion M and H in the form

$$M = r^2(\dot{\varphi} + \omega_0) = const, \tag{6}$$

$$H = \dot{r}^2 + \omega_0^2 r^2 + M^2/r^2 + 2\varepsilon U(r) = const. \tag{7}$$

Here $\omega_0 = eB_z/(2m\gamma)$, $\varepsilon = e/(m\gamma^3)$, e and m are charge and mass of a particle, B_z is $z-$component of the magnetic flux density, γ is the Lorentz factor.

Assume that function $\omega_0^2 r^2 + M^2/r^2 + 2\varepsilon U(r)$ is strictly convex. Then φ, M, and H define a particle trajectory. Therefore φ, M, H, and the phase of the trajectory θ can be taken as particle coordinates in the phase space. Additionally assume that particles are evenly distributed on phases of a trajectory. It means that trajectory segments corresponding to equal time intervals contain equal number of particles. This assumption provides the stationarity of the distribution.

As distribution is uniform on φ and θ, consider two-dimensional supspace with coordinates M and H as the

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phase space. Call this space the space of integrals of motion. Denote the single component of the phase density in these coordinates by $f(M,H)$. It is easy to understand that any $f(M,H)$ gives a solution of the Vlasov equation written in the form (2). To find $U(r)$ consider component of the phase density in the initial phase space in coordinates φ, θ, M, H

$$n_{\varphi\theta MH} = (4\pi)^{-1} f(M,H)/P(M,H), \tag{8}$$

where $P(M,H)$ is phase increment along half of a traejectory

$$P(M,H) = \int\limits_{r_{\min(M,H)}}^{r_{\max(M,H)}} (H - \omega_0^2 r^2 - M^2/r^2 - 2\varepsilon U(r))^{1/2}\, dr. \tag{9}$$

Find density $\varrho(r)$ in the configuration space. We have

$$n_{xyMH} = n_{\varphi\theta MH} \cdot det\left(\frac{\partial(\varphi,\theta)}{\partial(x,y)}\right) = \frac{n_{\varphi\theta MH}}{r|\dot{r}|}. \tag{10}$$

Hence,

$$\varrho(r) = \frac{1}{2\pi r}\int\limits_{\Omega(r)} \frac{P(M,H)^{-1} f(M,H)\, dM\, dH}{(H - M^2/r^2 - \omega_0^2 r^2 - 2\varepsilon U(r))^{1/2}}. \tag{11}$$

Here $\Omega(r)$ is the set of admissible values of M and H for particles passing through point with coordinate r.

It can be shown that if the beam is radially confined, $r \le R$, then $\Omega(r)$ is defined by the inequalities

$$\frac{M^2}{r^2} + \omega_0^2 r^2 + 2\varepsilon U(r) \le H \le \frac{M^2}{R^2} + \omega_0^2 R^2 + 2\varepsilon U(R). \tag{12}$$

Denote the set of all admissible values of M and H by Ω_R. It is defined by the inequalities

$$\min_r(\omega_0^2 r^2 + M^2/r^2 + 2\varepsilon U(r)) < H \le \tag{13}$$

$$\le M^2/R^2 + \omega_0^2 R^2 + 2\varepsilon U(R). \tag{14}$$

UNIFORMLY CHARGED BEAM

Let us find such phase distributions that particle density in the configuration space is uniform inside the beam cross-section $\varrho_{xy}(r) = \varrho_0$, $r \le R$. Then the Poisson equation yields $U(r) = -e\varrho_0 r^2/4\varepsilon_0$.

Firstly, consider the case when particle are distributed on the two-dimensional surface $M = 0, H = 0$. As previously, assume that the phase density does not depend of φ and θ. Using the Vlasov equation in the form (3), we get

$$\frac{\partial n_{\varphi\theta}}{\partial t} = -\frac{d\varphi}{dt}\frac{\partial n_{\varphi\theta}}{\partial \varphi} - \frac{d\theta}{dt}\frac{\partial n_{\varphi\theta}}{\partial \theta} = 0. \tag{15}$$

This solution corresponds to wide known Brillouin flow [12], when particle rotates around beam axis with the same angular velocity $\dot{\varphi} = -\omega_0$. As can be seen from (7), $\varrho_B = 2\varepsilon_0\omega_0^2/(e\varepsilon) = \varepsilon_0 B_z\gamma/(2m)$ is the spatial density of the Brillouin flow. In what follows, it is assumed that $\varrho_0 < \varrho_B$.

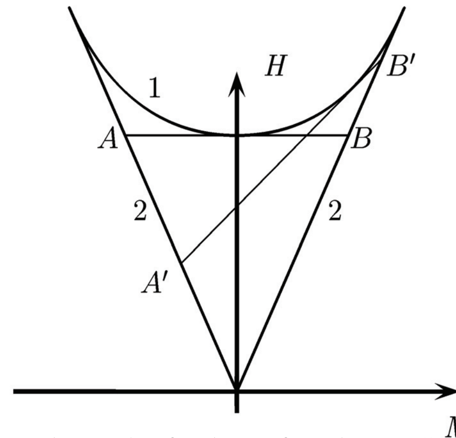

Figure 1: The set Ω_R for the uniform beam. Curve 1 represents the boundary $H = M^2/R^2 + \omega^2 R^2$. Straight line segments 2 represent the boundary $H = 2\omega|M|$.

If M and H can change, then inequalities (14) take the form

$$2\omega|M| < H \le M^2/R^2 + \omega^2 R^2, \tag{16}$$

where $\omega^2 = \omega_0^2 - e\varrho_0\varepsilon/(2\varepsilon_0)$. Consider a distribution when all particles are uniformly distributed on the straight line segment S_k, which is tangent to upper boundary of the set Ω_R :

$$S_k : H = kM + H_0, \quad H_0 = R^2(\omega^2 - k^2/4), \tag{17}$$

$|k| < 2\omega$, $(M,H) \in \Omega_R$ (segment $A'B'$ on Fig.1). In this case, the particle density in the space of integrals of motion is described by the differential form of degree 1 $f_0 dM$, $f_0 > 0$. In the initial phase space such density is described by the form of degree 3 defined on a three dimensional surface corresponding to segment S_k. Denote its components by n_{ijk}. Analogously to previous case, we get

$$n_{\varphi\theta M} = \frac{f_0}{4\pi P(M,H)}, \qquad n_{xyM} = \frac{n_{\varphi\theta M}}{r|\dot{r}|}. \tag{18}$$

Then spatial density does not depend of r :

$$\varrho_0 = 2\int\limits_{M_1}^{M_2} n_{xyM}\, dM = \frac{\omega f_0}{\pi} = const. \tag{19}$$

Here M_1, M_2 are roots of the denominator in the integrand.

At $k = 0$ (segment AB on fig.1), this distribution represents wide known Kapchinsky-Vladimirsky distribution [13], for which all particles are uniformly distributed on the segment AB (Fig.1).

All distribution corresponding to various k give uniformly charged beam with the same radius R. Therefore any linear combinations of these distributions or their integral on the parameter k give the uniform charged beam with radius R. As an example of nontrivial distribution gotten as integral, give the distribution with the density

$$f(M,H) = \frac{\pi\varrho_0}{2\omega^2(M^2 - HR^2 + \omega^2 R^4)^{1/2}}. \tag{20}$$

LONGWISE NONUNIFORM BEAM

Consider stationary azimuthally symmetric beam in longitudinal magnetic field in which all particles have the same longitudinal velocity v^z [7-14]. Let R and ω_0 slow change along beam axis: $d\omega_0/dz \ll \omega_0/R$. Assume also that the spatial density is uniform within each cross-section: $\varrho_{xy} = \varrho_0(z), r < R$.

In this case, M is also integral of motion. To get another integral, consider equation of radial motion

$$\frac{d^2r}{dt^2} = -\omega_0^2 r + \frac{M^2}{r^3} + \lambda \frac{r}{R^2} \quad (21)$$

and equation for the beam envelope $R(z)$

$$\frac{d^2R}{dt^2} = -\omega_0^2 R + \frac{\lambda}{R} + \frac{a_0^2 c_0^2}{R^3}, \quad (22)$$

which holds under assumption that at initial instance patricles lie inside the ellipse $r^2/a_0^2 + \dot{r}^2/c_0^2 = 1$ in the phase space of the transverse motion. Here $\lambda = eJ/(2\pi\varepsilon_0 m\gamma^3 v^z)$.

System of equations (21), (22) is similar to known Ermakov system [15] and generalized Ermakov system [16,17], but differs from them , because it contains terms with R^{-2} and R^{-1} in the first and in the second equations correspondingly. Integral of this system is

$$I = (\frac{dq}{d\tau})^2 + \frac{M^2}{q^2} + a_0^2 c_0^2 q^2. \quad (23)$$

Here $q = r/R, d\tau = ds/R^2$. Denote the set of admissible values of M and I by $\tilde{\Omega}_1$. It is easy to see that $\tilde{\Omega}_1$ is determined by inequalities

$$2a_0c_0|M| < I \leq M^2 + a_0^2 c_0^2, \quad (24)$$

and, therefore, looks like the set Ω_R for radially confined beam on Fig.1, where H should be replaced by I.

Consider a particle distribution of some thin layer moving along beam axis. The phase space is four-dimensional, and M, I, φ and θ can be taken as coordinates. As previously, assume that particle uniformly distributed on phases θ and azimuthal angle φ.

At first, consider a case when particles are distributed on the two-dimensional surface $M = 0, I = 0$. Equation (3) also yields equality (15). Therefore, such distribution is stationary solution of the Vlasov equation. From physical point of view, it correspond to a beam with radius changing along beam axis according to equation (22), and rotating in each cross-section with angular velocity that also depends on z. Such distribution is analogue to the Brillouin flow, and can be called the generalized Brillouin flow.

Consider also a distribution when all particles are uniformly distributed on the segment S_k, which is tangent to upper boundary of the set $\tilde{\Omega}_1$:

$$S_k : I = kM + I_0, \quad I_0(k) = a_0^2 c_0^2 - k^2/4, \quad (25)$$

$|k| < 2a_0c_0$, $(M,I) \in \tilde{\Omega}_1$ (segment $A'B'$ on Fig.1). Describe the particle density in the space of the integrals of motion by the differential form of the first degree $f_0 dM$, $f_0 > 0$. In the initial four-dimensional phase space such density is described by the form of degree 3 defined on the segment S_k. Analogously to the previous case, we get

$$n_{\varphi\theta M} = \frac{f_0}{4\pi P(M,I)}, \qquad n_{\tilde{x}\tilde{y}M} = \frac{n_{\varphi\theta M}}{q|\dot{q}|}, \quad (26)$$

where $\tilde{x} = x/R, \tilde{y} = y/R$,

$$P(M,I) = \int\limits_{q_{\min(M,I)}}^{q_{\max(M,I)}} (I - \frac{M^2}{q^2} - a_0^2 c_0^2 q^2)^{1/2}\, dq = \frac{\pi}{2a_0c_0}. \quad (27)$$

For spatial density we get

$$\varrho_{\tilde{x}\tilde{y}M} = \int\limits_{M_1}^{M_2} n_{\tilde{x}\tilde{y}M}\, dM = \frac{a_0c_0f_0}{\pi} = const. \quad (28)$$

When $k = 0$ (segment AB on Fig.1), we have analogue of the Kapchinsky-Vladimirsky distribution for nonuniform beam. It is easy to understand that taking a linear combination of such distributions with various k we also get a solution of the Vlasov equation.

Analogous approach can be also used for beam in extermal electric field [2, 18].

Analytical solutions of the Vlasov equation described here and others can be used as a beam models in optimization problems [19-25] and as test problems for beam simulation software.

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