ADVANCES IN IMPEDANCE THEORY*

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Abstract

We review recent progress in the following areas of the impedance theory: calculation of impedance of tapers and small angle collimators; optical approximation and parabolic equation for the high-frequency impedance; impedance due to resistive inserts in a perfectly conducting pipe.

INTRODUCTION

A remarkable progress over the last decade in development of computer codes significantly advanced our capabilities in calculation of wakefields and impedances for accelerators. There are however a number of practical problems that, when approached numerically, require a huge mesh, and hence memory, or an extraordinary CPU power, or both. One class of such problems is related to wakes of ultra short bunches, typical for many next generation electron/positron accelerators and photon sources. Another class is represented by long tapered transitions, often with non-round cross sections.

The numerical difficulties associated with these problems can be traced to a small parameter in the system, such as, e.g., a ratio of the bunch length to the length of a taper. It is remarkable, however, that the same small parameter can often be used to develop approximate analytical methods that provide a simplified solution to the impedance problem.

In this paper, we review recent results in the analytical theory of wakefields, which include long tapered transitions, calculation of the wakes for very short bunches, and some special cases of the resistive wall impedance.

TRANSVERSE IMPEDANCE OF TAPERED TRANSITIONS

Computer simulations of tapers are not always easy to carry out, especially in cases when the taper cross section is strongly elongated in the horizontal direction. Several analytical approaches has been developed in the past to treat gradual tapers.

Yokoya derived both the longitudinal and transverse impedances of a smooth axisymmetric transition in the low-frequency approximation [1]. His result for the transverse impedance is given by the following equation

$$Z_\perp = -\frac{iZ_0}{2\pi}\int dz\left(\frac{a'}{a}\right)^2 , \tag{1}$$

where $Z_0 = 4\pi/c = 377$ Ohm, $a = a(z)$ is the radius of the pipe, and the prime denotes derivative with respect to the axial coordinate z. Applying this formula to a conical transition with the conical angle θ that connects two pipes of radii a_1 and a_2 ($a_2 > a_1$) gives the following result:

$$Z_\perp = -\frac{iZ_0}{\pi a_{\rm av}}\frac{\epsilon\tan\theta}{1-\epsilon^2} , \tag{2}$$

where $a_{\rm av} = (a_1 + a_2)/2$ and $\epsilon = (a_2 - a_1)/(a_2 + a_1)$. The result (2) should be considered as a first approximation in the small parameter θ, and one can raise a question if (2) can be improved by adding higher order terms in the angle θ [2]. Such terms can be systematically derived for a smooth taper—they involve higher order derivatives of a with respect to z. Unfortunately, these terms diverge for a conical collimator for which a'' is proportional to the delta functions located at the transition points between the straight pipes and the conical taper.

By way of ingenious summation of diverging terms in an infinite sum of a perturbation theory Podobedov and Krinsky [2] found a leading order correction to (2),

$$Z_\perp = -\frac{iZ_0}{\pi a_{\rm av}}\frac{\epsilon\tan\theta}{1-\epsilon^2}\left(1-\frac{0.18}{\epsilon}\tan\theta\right) , \tag{3}$$

which was also confirmed by numerical simulations. Using several fitting parameters, they also found interpolation formulas for the impedance of the taper that provide an excellent approximation for $0 < \theta < \pi/2$ and $0 < \epsilon < 1$.

The original Yokoya derivation was simplified and extended to the case of rectangular geometry in Refs. [3–5], see Fig.1a. It was also found in Ref. [4] that, in the limit

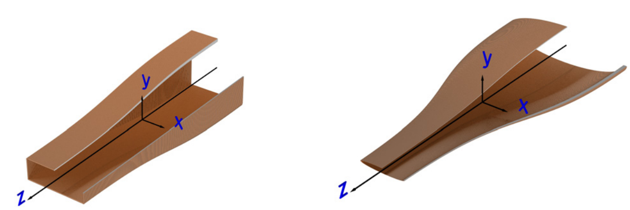

Figure 1: Smooth tapers with rectangular (*a*) and elliptic (*b*) cross sections.

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of small frequencies, the impedance of a rectangular transition increases linearly with the width of the taper when the ratio of the width of the transition to the gap becomes large. The dipolar component of the vertical impedance for a taper with rectangular cross section, was found to be [5]

$$Z_{\perp y} = -\frac{i}{4} w Z_0 \int dz \frac{(g')^2}{g^3} , \quad (4)$$

where w is the (constant) width in the x direction and $g = g(z)$ is the (varying) gap of the taper in the y direction. A comparison of (1) and (4) shows that impedance of a rectangular transition is roughly w/g times larger than that of a round one with the same length l.

Podobedov and Krinsky [6] calculated the impedance of a taper with confocal elliptical cross sections (see Fig.1b) and confirmed a similar scaling for the vertical dipolar impedance with the width of the ellipse, in the limit of large aspect ratios. More precisely, they obtained for the vertical dipolar impedance

$$Z_{\perp y} = -\frac{i\pi}{16} Z_0 \int dz \frac{(\rho')^2}{\rho^3} , \quad (5)$$

where $\rho(z) = \tanh^{-1}[b(z)/a(z)]$ with a and b the major (horizontal) and minor (vertical) semiaxes of the elliptical cross section, respectively. In the limit of large aspect ratio, $a \gg b$, it follows from this formula that $Z_{\perp y} \propto a$.

There is an important question of what are conditions of applicability of the results presented above. There are two such conditions—one is geometric, and the other one is related to the requirement of a small frequency ω. The geometric condition for the round geometry is $a \ll l$, or a small angle taper, $\theta \sim a/l \ll 1$. For a large aspect ratio rectangular transition the requirement is $g \ll w \ll l$, as was pointed out in [6], and similarly $b \ll a \ll l$ for the elliptic case. The regime $g \ll l \ll w$ for which the width of the collimator is wider than its length, has been addressed by Krinsky [7] in the limit when the relative change in the vertical gap through the transition is small, $\Delta g/g \ll 1$. The results of that paper indicate that the vertical impedance as a function of width w saturates at $w \sim l$ and does not increase with the width when w becomes larger than l.

As was mentioned above, the results are valid at low frequencies or, equivalently, for long bunches. To evaluate the effect of bunch length on the impedance in a simple unifying manner it is useful to consider a kick factor due to the transition. The kick factor κ_y (in the vertical direction) is defined by

$$\kappa_y = \frac{E}{eQ} \frac{\langle \theta_y \rangle}{\langle y \rangle} , \quad (6)$$

where Q and E are the bunch charge and energy, respectively, $\langle y \rangle$ is the averaged offset of the beam in the taper, and $\langle \theta_y \rangle$ is the averaged over the beam the kick angle due to the impedance. For a purely inductive impedance (represented by Eqs. (1)-(5)) and a Gaussian bunch, the kick factor is given by

$$\kappa_y = -\frac{c}{2\sqrt{\pi}\sigma_z} \mathrm{Im} Z_{\perp y} , \quad (7)$$

and it scales inversely proportionally to the rms bunch length σ_z. For shorter bunches, the impedance is not purely inductive, and the scaling of the kick factor with the bunch length deviates from $\kappa_y \propto \sigma_z^{-1}$.

For the round geometry, the frequency range where one can use Eq. (1) is well known and is given by $ka^2 \ll l$, where $k = \omega/c$ and a is understood as a characteristic value of the radius. In the opposite limit of high frequecies, $ka^2 \gg l$, the kick factor does not depend on the bunch length.

For large aspect ratio collimators the situation is more complicated. Theoretical analysis in Refs. [8, 9] showed that the inductive regime transitions at $kw^2/l \sim 1$ to an intermediate one with the scaling $\kappa_y \sim \sigma_z^{-1/2}$, which then transitions at $kg^2/l \sim 1$ to a high frequency regime where κ_y does not depend on σ_z.

Numerical simulation of Ref. [6] indeed revealed the intermediate regime. Based on numerical simulations however, Podobedov and Krinsky conjectured that the transition to this regime actually occurs at $k \sim 1/w$. It is author's belief that the issue is still open and requires additional studies.

OPTICAL APPROXIMATION IN IMPEDANCE THEORY

Calculation of impedance for extremely short bunches such as those envisioned for the International Linear Collider [10] with the nominal bunch length of $\sigma_z = 300$ microns, or in the Linac Coherent Light Source where the bunch length can be as short as a few microns [11], introduces new possibilities to the theory. The frequency spectrum of the bunch is characterized by $k \sim \sigma_z^{-1}$ and the small parameter for the problem is $(kb)^{-1}$ with b the characteristic geometrical size of an object in the vacuum pipe that generates the impedance. It has long been known that effective utilization of this small parameter may lead to simplification of the impedance problem, and several analytical results are available in the literature for the impedance at high-frequencies. They include the impedance of a step transition [12, 13] and the diffraction model for the impedance of a cylindrical pillbox cavity [14, 15].

A general method which allows one to calculate the impedance in the limit of very high frequencies was recently developed in Refs. [16, 17] under the name of the *optical approximation* (or *optical regime*) in the theory of impedance. In this approximation it is assumed that the electromagnetic fields carried by a short bunch propagate along straight lines equivalent to rays in the geometric optics. A protrusion or an obstacle inside the beam pipe can intercept the rays and reflect them away from their original direction. The energy in the reflected rays is associated with the energy radiated by the beam, which can then be related to the impedance. Note that this kind of argument has been used in the past in the case of step-in and step-out transitions in a round pipe, where the impedance was related to the energy "clipped away" from the beam by the step [18–20].

If l is the length of an obstacle and b is the minimal cross-section size of the beam pipe, the conditions for the optical approximation are

$$1 \ll kb\,, \qquad l \ll kb^2\,. \tag{8}$$

The first of these two conditions requires the size of the obstacle be much larger than the reduced wavelength of the radiation. The right hand side of the second inequality has a meaning of the length over which diffraction effects become significant, and this relation guarantees that such effects give only a small correction to those of geometric optics. Note that even a small-angle taper of angle $\theta \sim b/l$ can be described in the optical approximation for short enough bunches, if $\sigma_z/b \ll \theta$. The quantity kb^2 can also be interpreted as a catch-up distance over which radiation, generated by the head of a beam at the lateral surface at distance b from the beam orbit reaches the beam tail at distance σ_z behind the head. Thus the second condition of Eq. (8) for the applicability of the optical approximation is that the object is short compared to the catch-up distance.

The result of Refs. [16, 17] can be formulated as follows. Consider a transition from a cylindrical pipe of a given cross section S_A to a pipe of cross section S_B with $S_{\rm ap}$ being the minimal cross-section of the transition. An example of such three-dimensional transition is shown in Fig. 2. Let us denote $\boldsymbol{r}_1 = (x_1, y_1)$ the two-dimensional

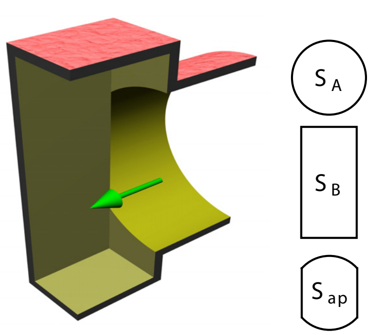

Figure 2: Cutout of the transition from a round to rectangular pipe. The arrow shows the direction in which the beam travels. The shapes on the right side show cross sections of the incoming, S_A, and outgoing, S_B, pipes, as well as the minimal cross section for the transition $S_{\rm ap}$.

position of the leading point charge q_1 and $\boldsymbol{r}_2 = (x_2, y_2)$ the two-dimensional position of the trailing point charge, at distance s behind. The longitudinal wake $w_\parallel(\boldsymbol{r}_1, \boldsymbol{r}_2, s)$ is then defined by

$$w_\parallel(\boldsymbol{r}_1, \boldsymbol{r}_2, s) = -\frac{c}{q_1}\int_{-\infty}^{\infty} dt E_{1,z}(\boldsymbol{r}_2, z = ct - s, t)\,,$$

where $E_{1,z}$ is the longitudinal component of the electric field generated by the first charge at the position of the second one.

The optical approximation for the wake gives a purely resistive wake field proportional to the delta function:

$$w_\parallel(\boldsymbol{r}_1, \boldsymbol{r}_2, s) = \frac{1}{2\pi}\delta(s) I(\boldsymbol{r}_1, \boldsymbol{r}_2)\,, \tag{9}$$

with the corresponding longitudinal impedance

$$Z_\parallel(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{2\pi c} I(\boldsymbol{r}_1, \boldsymbol{r}_2)\,. \tag{10}$$

Note that the longitudinal impedance in this case does not depend on the frequency. The factor I in the above equations is

$$\begin{aligned} I = &\int_{S_B} \nabla\phi_{1,B}(\boldsymbol{r})\cdot\nabla\phi_{2,B}(\boldsymbol{r})\, dS - \\ &- \int_{S_{ap}} \nabla\phi_{1,A}(\boldsymbol{r})\cdot\nabla\phi_{2,B}(\boldsymbol{r})\, dS\,, \end{aligned} \tag{11}$$

with $\nabla = \hat{\boldsymbol{x}}\,\partial/\partial x + \hat{\boldsymbol{y}}\,\partial/\partial y$. The integration goes over the cross section of the outgoing pipe S_B and the minimal cross section S_{ap}, indicated as subscripts. The potential ϕ satisfies Poisson's equation with a delta function on the right hand side:

$$\begin{aligned} \nabla^2\phi_{1,B}(\boldsymbol{r}) &= -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}_1)\,, \\ \nabla^2\phi_{2,B}(\boldsymbol{r}) &= -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}_2)\,, \end{aligned} \tag{12}$$

with the boundary conditions $\phi_{1,B} = \phi_{2,B} = 0$ on the wall of pipe B (similar equations hold for $\phi_{1,A}$).

Knowledge of the longitudinal impedance (10) allows one to compute the transverse impedance using the Panofsky-Wenzel theorem

$$\boldsymbol{Z}_\perp = \frac{c}{\omega}\nabla_{\boldsymbol{r}_2} Z_\parallel\,.$$

Many examples of the calculation of the impedance in optical approximation, including geometries shown in Fig. 3, can be found in Ref. [17], as well as comparison with numerical simulations and the discussion of the accuracy of the model. A practical application of the optical theory for calculation of the wake due to coupler asymmetry in Tesla-type accelerator cavities for the ILC can be found in Ref. [21].

PARABOLIC EQUATION FOR ELECTROMAGNETIC FIELD

Diffraction phenomena lie beyond the limits of the optical approximation. They, however, can be accounted for in a simplified treatment based on a so called *parabolic equation* (PE) [22]. The parabolic equation in the diffraction theory was proposed many years ago [23] and has been widely used since that time. It is also a standard approximation in the FEL theory [24]. More recently, the paraxial

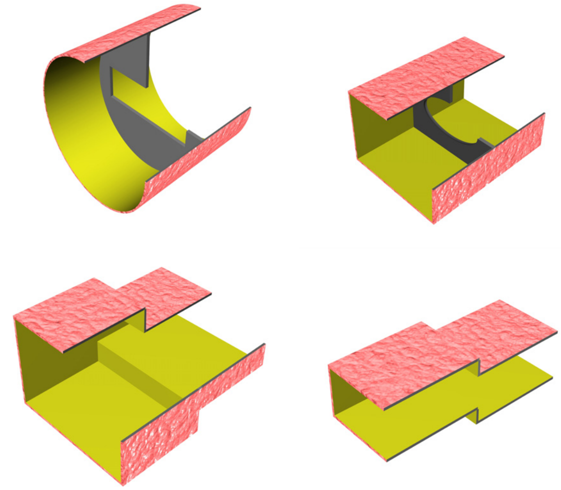

Figure 3: Examples of transitions solvable with the optical approximation.

approximation was applied to the beam radiation problems in a toroidal waveguide [25, 26] and in free space [27].

Applicability of PE in the high-frequency limit of the impedance is based on the observation that in this case the main contribution to the impedance comes from the electromagnetic waves that catch up with the beam far from the obstacle and propagate at small-angles to the axis of the pipe. This allows one to simplify Maxwell's equations using the *paraxial* approximation. The resulting PE in many cases is easier to solve than the original wave equations for the fields. Note that the parabolic equation does not require the second of the conditions in Eq. (8) to be satisfied, and can be applied to long tapers at high frequencies.

PE is formulated for the transverse (with respect to the direction of motion of the beam) component of the electric field $\hat{\boldsymbol{E}}_\perp$ where the hat denotes a Fourier transformed quantity,

$$\hat{\boldsymbol{E}}(x, y, z, \omega) = \int d\omega\, e^{i\omega t - ikz}\, \boldsymbol{E}(x, y, z, t).$$

The transverse component of the electric field $\hat{\boldsymbol{E}}_\perp$ is a two-dimensional vector $\hat{\boldsymbol{E}}_\perp = (\hat{E}_x, \hat{E}_y)$. The PE reads

$$\frac{\partial}{\partial z}\hat{\boldsymbol{E}}_\perp = \frac{i}{2k}\left(\nabla_\perp^2 \hat{\boldsymbol{E}}_\perp - \frac{4\pi}{c}\nabla_\perp \hat{j}_z\right), \qquad (13)$$

where $\nabla_\perp = (\partial/\partial x, \partial/\partial y)$ and $\hat{j}_z$ is the Fourier transformed projection of the beam current in the direction z. The longitudinal electric field can be expressed through the transverse one and the current

$$\hat{E}_z = \frac{i}{k}\left(\mathrm{div}\, \hat{\boldsymbol{E}}_\perp - \frac{4\pi}{c}\hat{j}_z\right). \qquad (14)$$

One of the most important advantages of PE is that it eliminates the small wavelength $2\pi/k$ from the problem. Indeed, a simple scaling analysis of (13) shows that the longitudinal scale l in that equation is of the order of ka^2, where a is the transverse size of the problem. As a result, the numerical solution of PE requires coarser spatial meshes.

As in illustration, we show in Fig. 4 the longitudinal impedance of a round tapered collimator calculated with PE up to the frequency of about 4 THz. The collimator has two tapered transitions of length 30 mm from radius of 5 mm to radius of 2.5 mm. It also has a central part (2.5 mm radius) of length of 30 mm. The result is compared with simulation with the computer code ECHO [28], and shows an excellent agreement. Note that

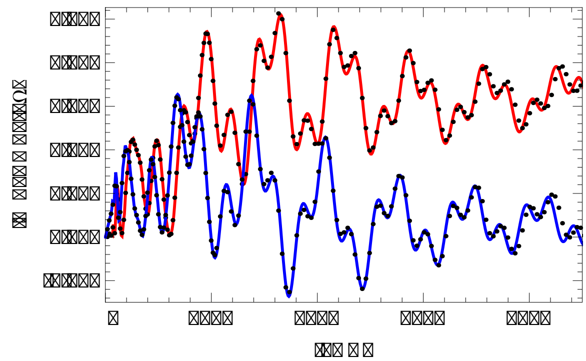

Figure 4: Real (red) and imaginary (blue) parts of the impedance. Dots are calculated with PE, solid lines are the result of the computer code ECHO.

the real part of the impedance at large frequencies approaches the value of about 90 Ohm, which is close to what is expected from the optical approximation for such collimator, $\mathrm{Re}Z = (Z_0/\pi)\ln(a_{\mathrm{max}}/a_{\mathrm{min}}) = 83$ Ohm.

IMPEDANCE OF RESISTIVE WALL INSERTS

Resistive wall impedance is one of the oldest subjects of the impedance theory [29]. Usually, however, this

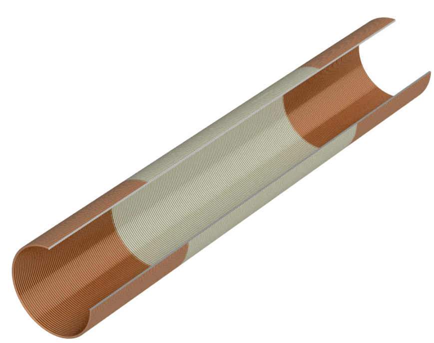

Figure 5: Cylindrical insert of higher resistivity in a round pipe.

impedance is calculated for an infinitely long pipe. There are situations in practice, when a pipe with a high wall conductivity has a short resistive insert of length L_i as shown in Fig. 5. Recent papers [30–32] address this problem.

We remind the reader that the resistive wall impedance per unit length of a long pipe is given by the following formula [29]

$$\frac{Z_\parallel}{L_i} = \frac{cZ_0}{4\pi}\frac{1-i}{ca}\sqrt{\frac{\omega}{2\pi\sigma}}\left[1-\frac{a\omega}{4c}(1+i)\sqrt{\frac{\omega}{2\pi\sigma}}\right]^{-1}. \quad (15)$$

In the limit of low frequencies, $ks_0 \ll 1$, where $k = \omega/c$ and $s_0 = (2a^2/Z_0\sigma)^{1/3}$ (σ is the wall conductivity, a is the pipe radius), one has

$$\frac{Z_\parallel}{L_i} = \frac{cZ_0}{4\pi}\frac{1-i}{ca}\sqrt{\frac{\omega}{2\pi\sigma}}. \quad (16)$$

In the opposite limit, $ks_0 \gg 1$, the impedance per unit length does not depend on conductivity and is $Z_\parallel/L_i = (cZ_0/4\pi)(4i/a^2\omega)$.

A naive approach to the case of an insert would be to multiply the impedance per unit length of an infinite pipe by the length of the insert L_i. This indeed turns out to be true in the limit of low frequencies, $ks_0 \ll 1$, as was shown in [31].

At high frequencies, when $ks_0 \gg 1$, the detailed analysis of Ref. [30] reveals several regimes. The authors show that one can use (15) for the impedance of long inserts, when $L_i \gg ka^2$. In an intermediate regime when $a^2/s_0^3k^2 \ll L_i \ll ka^2$ (note that the left hand side of this inequality is indeed much smaller than the right hand side because of assumed $ks_0 \gg 1$) the impedance does not depend on conductivity and is equal to twice the diffraction impedance of a pill-box cavity of length L_i [29]

$$Z_\parallel = \frac{Z_0(1-i)}{\pi a}\sqrt{\frac{cL_i}{\pi\omega}}. \quad (17)$$

Finally, for very short inserts, $a^2/s_0^3k^2 \gg L_i$, the impedance is again given by Eq. (16) multiplied by the length of the insert.

CONCLUSION

We reviewed recent analytical results in the theory of beam impedance in accelerators. They include: a low frequency approximations for long tapers of various cross sections; optical approximation that allows to compute geometric impedance of short bunches generated by protrusions into the vacuum chamber; the parabolic equation for simplified description of the electromagnetic field at high frequencies; and, finally, resistive wall impedance of short inserts.

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