# GRAVITATIONAL INSTABILITY OF A NONROTATING GALAXY*

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*Abstract*

Gravitational instability of the distribution of stars in a galaxy is a well-known phenomenon in astrophysics. This report is an attempt to analyze this phenomenon by applying standard tools developed in accelerator physics. It is found that a nonrotating galaxy would become unstable if its size exceeds a certain limit that depends on its mass density and its velocity spread.

## INTRODUCTION

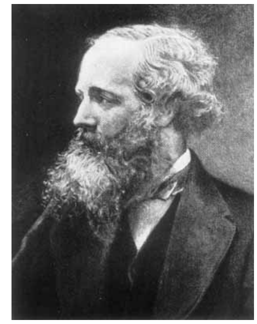
There are some notable examples in the past when developments in astrophysics are later found to be connected to important topics in accelerator physics. The two major topics in accelerator physics, nonlinear dynamics and collective effects, each has its origin traced back to astrophysics.

On nonlinear dynamics, Poincaré was believed to be the first to note the behavior of chaos. In 1887, he entered a contest sponsored by the king of Sweden and Norway, and the problem was to prove that the solar system as a three-body system was dynamically stable. He failed to prove it, but his work won the prize. Poincaré also introduced the concept of Poincaré section, which accelerator physicists use everyday as they try to describe the turn-by-turn motion of single-particles in synchrotrons and storage rings. Indeed, what a beam position monitor detects in these circular accelerators is a special case of Poincaré section. Dynamic aperture and chaotic motion are also typically observed as Poincaré sections.

Henri Poincaré (1854-1912)

On collective effects, one notable preview from astrophysics was the impressive work by Maxwell. In 1857, Maxwell won the Adams Prize when he proved analytically that the Saturn ring can not be stable unless it consists of many small satellites instead of a single solid piece. Today, we call this mechanism of Maxwell "negative mass instability" in accelerator physics.

Following these ground-breaking pioneers, one might ask if today, after years of evolution, might there be some accelerator physics studies that can be applied to astrophysics in return. One such attempt is ventured here. We will try to apply modern accelerator techniques [1] to the well-known problem [2] of a gravitational instability of a galaxy. If this approach turns out fruitful, a large arsenal of analysis tools can potentially be transported from accelerator physics to this and other problems in astrophysics.



James Clerk Maxwell (1831-1879)

The fact that there is a gravitational instability is rather obvious. Consider a uniform distribution of stars in an infinite space. As a first picture, let all stars be initially stationary in space. Now consider a statistical fluctuation of the star distribution so that there is a slight excess of stars in a small region of space. This excess of stars generates an inward gravitational pull on the surrounding stars, yielding an increase of this excess as the stars begin to move. The initial small excess therefore grows, leading to an instability.

As will be seen later, this instability is counteracted by a spread in the stars' initial velocities. This spread of velocities can be represented by a "temperature" or a "pressure" of the galaxy. Its net effect is to counteract the gravitational instability and stabilize the galaxy under favorable conditions. In accelerator physics, this stabilizing mechanism is attributed to Landau damping.

Consider a distribution of stars in a galaxy described by a mass-density distribution $\rho(\vec{x},\vec{v},t)$ in the phase space $(\vec{x},\vec{v})$ at time $t$. We wish to analyze the stability of this distribution of stars under the influence of their collective gravitational force. To simplify the problem, we will use a flat Euclidean space-time and will consider Newtonian, nonrelativistic dynamics only. In other words, we ignore both the special theory and the general theory of relativity.

## DISPERSION RELATION

Consider a particular star in the galaxy. The equations of motion of this star are

$$\begin{aligned}\dot{\vec{x}} &= \vec{v} \\ \dot{\vec{v}} &= G\int d\vec{v}'\int d\vec{x}'\,\rho(\vec{x}',\vec{v}',t)\,\frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3}\end{aligned} \tag{1}$$

where $G$ is the gravitational constant. These equations do not depend on the mass of the star under consideration. Whether it is a star or a dust particle does not matter.

* Work supported by Department of Energy Contract No. DE-AC02-76SF00515

Following standard treatment in accelerator physics, evolution of $\rho$ is described by the Vlasov equation [3]

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial\vec{x}}\cdot\dot{\vec{x}}+\frac{\partial\rho}{\partial\vec{v}}\cdot\dot{\vec{v}}\\ =\ &\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial\vec{x}}\cdot\vec{v}+\frac{\partial\rho}{\partial\vec{v}}\cdot G\int d\vec{v}'\int d\vec{x}'\rho(\vec{x}',\vec{v}',t)\,\frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3}\\ =\ &0 \end{aligned} \qquad (2)$$

To examine the stability of the system, let the galaxy distribution be given by an unperturbed distribution $\rho_0$ plus some small perturbation. Let the unperturbed distribution $\rho_0$ depend only on $\vec{v}$,

$$\rho_0=\rho_0(\vec{v}) \qquad (3)$$

This unperturbed distribution is uniform in the infinite 3-D space. The function $\rho_0(\vec{v})$ is so far unrestricted. On the other hand, we allow the small perturbation around $\rho_0$ to depend on $t$ and $\vec{x}$. For the galaxy to be stable, the deviation must not grow in time for all possible initial deviations.

We Fourier decompose the perturbation and write

$$\rho(\vec{x},\vec{v},t)=\rho_0(\vec{v})+\Delta\rho(\vec{v})\,e^{-i\omega t+i\vec{k}\cdot\vec{x}} \qquad (4)$$

where $\vec{k}$ is the wavenumber vector and $\omega$ is the angular frequency of the perturbation. We anticipate that for a given $\vec{k}$ (real), there will be a specific solution for $\omega$ (complex). The imaginary part of $\omega$ is the instability growth rate (growth rate if Im$(\omega)>0$, damping rate if Im$(\omega)<0$). Our job is to find $\omega(\vec{k})$ as a function of $\vec{k}$. If we find for any $\vec{k}$ that its corresponding Im$\omega(\vec{k})>0$, the galaxy is unstable.

Substituting Eq.(4) into Eq.(2) and keeping only first order in $\Delta\rho$ yield

$$-i(\omega-\vec{v}\cdot\vec{k})\Delta\rho(\vec{v})+G\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})=0 \qquad (5)$$

where $\vec{q}$ is the Fourier transform of the Newton kernel $\vec{x}/|\vec{x}|^3$, and might be called the "graviton propagator" following a terminology in quantum field theory. In fact,

$$\vec{q}(\vec{k})=\frac{4\pi i}{|\vec{k}|^2}\,\vec{k} \qquad (6)$$

In accelerator physics, the Newton kernel $\vec{x}/|\vec{x}|^3$ stands for the wake function while its Fourier transform $\vec{q}$ stands for the impedance. A comparison of the languages used in these different fields looks like Table 1.

Equation (5) can be rewritten as

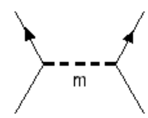
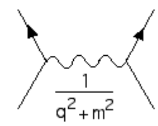
$$\Delta\rho(\vec{v})=-iG\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})}{\omega-\vec{v}\cdot\vec{k}} \qquad (7)$$

Integrating both sides over $\vec{v}$ and canceling out the common factor of $\int d\vec{v}'\,\Delta\rho(\vec{v}')$ then gives a dispersion relation that must be satisfied by $\omega$ and $\vec{k}$,

$$1=-iG\int d\vec{v}\frac{\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})}{\omega-\vec{v}\cdot\vec{k}} \qquad (8)$$

Given $\rho_0(\vec{v})$, we solve this dispersion relation for $\omega$ as a function of $\vec{k}$.

Table 1: Comparison of different languages.

| | kernel in coordinate space | propagator in momentum space |
|---|---|---|
| gravitational instability | $\frac{\vec{x}}{\lvert\vec{x}\rvert^3}$ | $\frac{4\pi i\vec{k}}{\lvert\vec{k}\rvert^2}$ |
| accelerator physics | wakefields $W_\parallel(z), W_\perp(z)$ | impedances $Z_\parallel(\omega), Z_\perp(\omega)$ |
| quantum field theory | exchange gauge particles   | propagators   |

## UNIFORM ISOTROPIC GALAXY

We next consider an unperturbed distribution that depends only on the magnitude of $\vec{v}$, i.e., let

$$\rho_0=\rho_0(|\vec{v}|^2) \qquad (9)$$

This is a uniform isotropic (uniform in $\vec{x}$, isotropic in $\vec{v}$) galaxy. Normalization condition is $\int_0^\infty 4\pi v^2 dv\rho_0(v^2)=\rho_m=$ mass density of stars per unit volume.

Substituting Eqs.(6) and (9) into Eq.(8) then gives

$$1=\frac{8\pi G}{|\vec{k}|^2}\int d\vec{v}\,\rho_0'(|\vec{v}|^2)\frac{\vec{v}\cdot\vec{k}}{\omega-\vec{v}\cdot\vec{k}} \qquad (10)$$

One must refrain from performing the integration over $\vec{v}$ at this time because that integral involves a singularity. Proper treatment of the singularity follows that of Landau damping [4], a general phenomenon that occurs in several branches of physics. The treatment amounts to adding an infinitesimal positive imaginary part to $\omega$, i.e. $\omega\to\omega+i\epsilon$. The integral around the singularity,

$$\int du\,\frac{1}{u-\omega}\ \to\ \int du\,\frac{1}{u-\omega-i\epsilon}$$

becomes an integration along the contour $\mathcal{C}_1$ as in Fig.1(a).

Using complex variable analysis, we know we can deform the integration contour from $\mathcal{C}_1$ to $\mathcal{C}_2$, as shown in

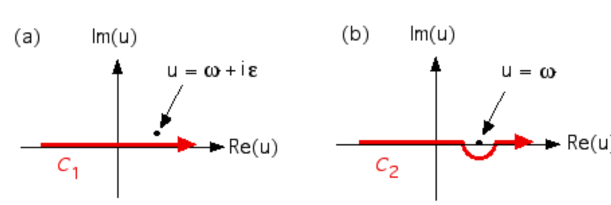



Figure 1: Integration contour dictated by Landau damping analysis.

Fig.1(b). Around the singularity pole, the contour traces out a perfect, left-right symmetric, infinitesimally small, half-circle. The integral then contains two terms, a principal value term from integration along the real axis of $\mathcal{C}_2$ and a residue term from the contribution of the half-circle:

$$\int du \frac{1}{u-\omega} \rightarrow \text{P.V.}\left(\int du \frac{1}{u-\omega}\right) - i\pi \times (\text{residue})$$

The P.V. is well-defined and has no singularity. Because of the residue term, the integral, which seems to be real in first appearance, actually contains an imaginary part! This is Landau damping mechanism in action.

To be specific, we next take a uniform distribution of $\rho_0$,

$$\rho_0(v^2) = \begin{cases} \frac{3\rho_m}{4\pi v_0^3} & \text{if } v^2 < v_0^2 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

This distribution is analogous to the "waterbag model" in accelerator physics. The quantity $v_0^2$ is related to the "temperature" of the galaxy. The dispersion relation now becomes

$$\lambda = \frac{1}{2 + x \ln\left|\frac{x-1}{x+1}\right| + i\pi x\, H(1-|x|)} \quad (12)$$

where

$$\lambda = \frac{6\pi G\rho_m}{k^2 v_0^2} \quad \text{and} \quad x = \frac{\omega}{k v_0} \quad (13)$$

In accelerator physics, $\lambda$ is replaced by the impedance. One simplification for the gravitational instability is that $\lambda$ is a real quantity, while the impedance is complex in general.

## STABILITY CONDITION

We next need to compute the instability growth rate, i.e. Im$\omega$ as a function of $k$. The star distribution $\rho_0(\vec{v})$ would be unstable if, for any $\vec{k}$, its corresponding $\omega$ has a positive imaginary part. We need to compute $x$ as a function of $\lambda$ using Eq.(12) in order to obtain $\omega$ as a function of $k$. Unfortunately Eq.(12) gives $\lambda$ as a function of $x$, and its inversion to give $x$ as a function of $\lambda$ is difficult. To deal with this difficulty, we apply another technique of accelerator physics as follows.

In general $x$ is complex, but at the edge of instability when the system is barely unstable, $x$ is real. The edge of instability can therefore be seen by plotting the right-hand-side of Eq.(12) as $x$ is scanned along the real axis from $-\infty$ to $\infty$. Fig.2 shows the result of such a scan. The horizontal and vertical axes are the real and imaginary parts of the right-hand-side of Eq.(12) respectively. As $x$ is scanned from $-\infty$ to $\infty$, the right-hand-side of Eq.(12) traces out a cherry-shaped diagram, including the "stem" of the cherry running from $-\infty$ to 0 along the real axis. The cherry curve defines the boundary between stable and unstable regions. If $\lambda$, the left-hand-side of Eq.(12), lies inside this cherry diagram (including the stem), the galaxy

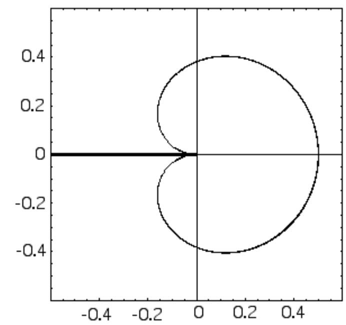



Figure 2: Stability diagram for the galaxy distribution.

distribution is stable. If it lies outside, the galaxy is unstable. Since $\lambda$ is necessarily real and positive, the stability condition therefore reads

$$\lambda < \frac{1}{2} \quad (14)$$

Equation (14) indicates that a hot galaxy (high temperature, i.e. large $v_0$) is more stable than a cold one. This is expected due to the Landau damping mechanism. It also indicates that the star distribution is most unstable for long-wavelength perturbations (small $k$). The threshold wavelength is given by

$$x_{\rm th} = \frac{2\pi}{k_{\rm th}} \quad \text{where} \quad k_{\rm th} = \frac{\sqrt{12\pi G\rho_m}}{v_0} \quad (15)$$

Perturbations with wavelength longer than $x_{\rm th}$ are unstable. One might expect that the galaxy will have a dimension of the order of $x_{\rm th}$ because if the galaxy had a larger dimension, it would have broken up due to the instability until it is reduced to the stable size.

It might be instructive to relate $v_0$ to an internal "pressure" and a "temperature" of the galaxy distribution,

$$P = \frac{2}{3}\rho_m\langle v^2\rangle = \frac{2}{5}\rho_m v_0^2, \quad T = \frac{m}{k_B}\frac{P}{\rho_m} \quad (16)$$

where $k_B$ is the Boltzmann constant.

## INSTABILITY GROWTH RATE

When $\lambda > 1/2$, $\omega$ will be complex with a positive imaginary part. The instability growth rate is given by $\tau^{-1} = \text{Im}(\omega)$. We need to find $\tau^{-1}$ as a function of $k$. To do so, we first scale the variables by

$$u = \frac{k v_0}{\sqrt{6\pi G\rho_m}}, \quad v = \frac{\tau^{-1}}{\sqrt{6\pi G\rho_m}} \quad (17)$$

and then

$$\frac{1}{u^2} = \frac{1}{2 - 2\left(\frac{v}{u}\right)\tan^{-1}\left(\frac{u}{v}\right)} \quad (18)$$

Fig.3 shows the result.

As seen from Fig.3, the growth rate vanishes ($v = 0$) when $u = \sqrt{2}$, corresponding to $\lambda = 1/2$, i.e. at the instability boundary. Figure 3 also shows that instability occurs

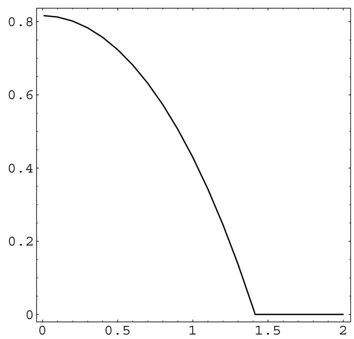



Figure 3: $v$ vs $u$ according to Eq.(18).

fastest for small $u$, i.e. small $k$ or long-wavelength perturbations. The growth rate at $u = 0$ has $v = \sqrt{2/3}$, or

$$(\tau^{-1})_{\max} = \sqrt{4\pi G \rho_m} \tag{19}$$

The result that fastest instability occurs for perturbations of infinitely long wavelength ($k = 0$) depends on our assumption of Newtonian dynamics of action-at-a-distance. Under this assumption, perturbation at one location instantly affects locations infinitely far away. If this assumption is appropriately removed, it is expected that the instability of long-wavelength perturbations will be weakened.

With the condition $\lambda < \frac{1}{2}$, stable galaxies must have a dimension smaller than a critical value, i.e.

$$\text{galaxy dimension} < \frac{2\pi v_0}{\sqrt{12\pi G \rho_m}} \tag{20}$$

Stability is provided through Landau damping. When the velocity spread $v_0 \to 0$, no galaxies can be stable. Equations (19) and (20) are our main results. In terms of the galaxy pressure, stability requires

$$(\text{galaxy dimention}) \times \rho_m < \sqrt{\frac{5\pi}{6}\frac{P}{G}} \tag{21}$$

Figure 4 shows four traces, each is the locus of the stability contour when $x$ is scanned from $-\infty$ to $\infty$ while $y$ is held fixed ($\frac{\omega}{k v_0} = x + iy$, with $y$ representing a nonzero growth rate). The four traces correspond, starting with the inner most one, to $y = 0.001, 0.01, 0.1, 0.3$. When $y = 0$, the trace reproduces Fig.2.

## OTHER DISTRIBUTIONS

So far we have assumed a somewhat nonrealistic water-bag distribution (11) for the galaxy's temperature. A few other examples are given below:

*Case 1:* $\rho_0(v^2) = \frac{\rho_m v_0}{\pi^2 (v^2+v_0^2)^2}$

*Case 2:* $\rho_0(v^2) = \frac{3\rho_m v_0^2}{4\pi (v^2+v_0^2)^{5/2}}$

*Case 3:* $\rho_0(v^2) = \frac{4\rho_m v_0^3}{\pi^2 (v^2+v_0^2)^3}$

*Case 4:* $\rho_0(v^2) = \frac{\rho_m}{4\pi\sqrt{2\pi}\, v_0^3} e^{-v^2/2v_0^2}$

Figure 5 shows the stability diagrams for the various cases. For the galaxy to be stable, the parameter $\lambda$ must be small than $\frac{3}{2}, \frac{3}{4}, \frac{1}{2}$ and 3, for cases 1, 2, 3, 4, respectively.

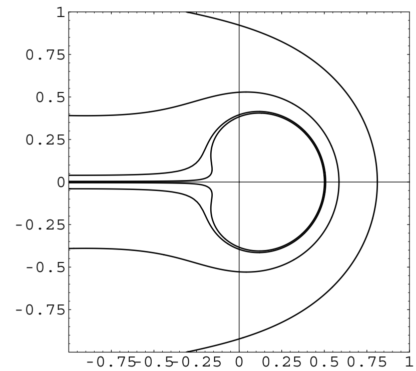



Figure 4: Contours of constant growth rates.

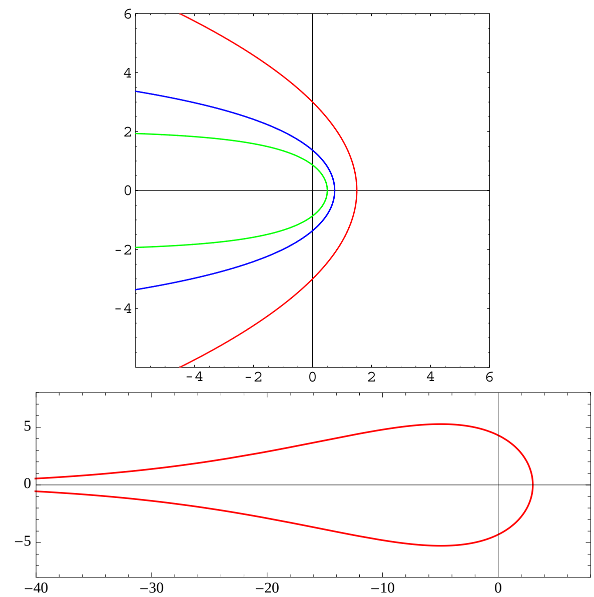



Figure 5: Upper: Stability diagrams for galaxy distributions are plotted in red, blue, green for Cases 1, 2, 3, respectively. Lower: Case 4.

## NUMERICAL ESTIMATES AND DISCUSSIONS

For a numerical application, we take estimates from the Milky Way [2, 5],

$$\rho_m = 2 \times 10^{-23}\ \mathrm{g/cm^3}, \quad v_0 = 200\ \mathrm{km/s}$$

Using the result of the water-bag model, we obtain a shortest growth time of $7 \times 10^6$ years for perturbations with very long wavelengths. For stability, the galaxy dimension must be smaller than 14000 light-years, which seems to be consistent with the size of the Milky Way.

It is conceivable that the same analysis can be applied to the dynamics of galaxies in a galaxy cluster, instead of stars in a galaxy. In that case, $\rho(\vec{x}, \vec{v}, t)$ describes the distribution of galaxies in the galaxy cluster. We might then take

$$\rho_m = 10^{-28}\ \mathrm{g/cm^3}, \quad v_0 = 1000\ \mathrm{km/s}$$

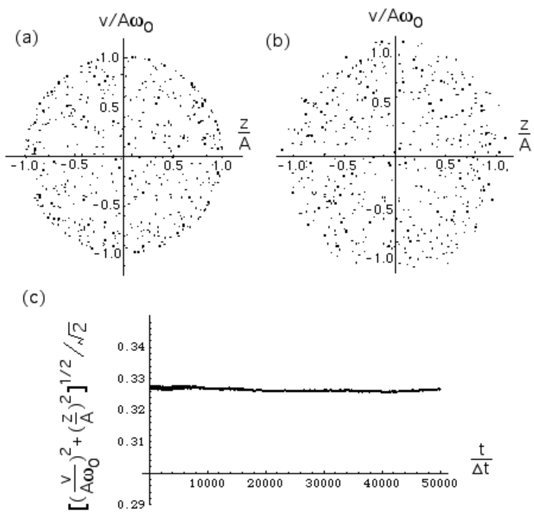



Figure 6: Simulation result of a 1-D nonrotating galaxy model. (a) Initial phase space distribution of 500 stars, (b) Final distribution after $2000/2\pi$ periods, (c) Plot of galaxy emittance $\sqrt{[(v/\omega_0 A)^2+(z/A)^2]/2}$ as a function of the step number $t/\Delta t$, showing no sign of instability.

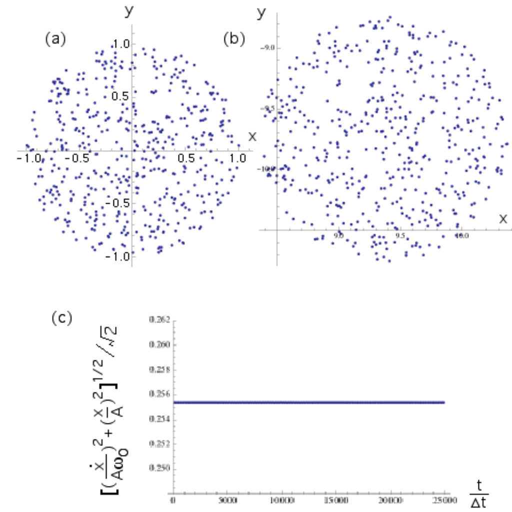



Figure 7: Simulation result of the 2-D rotating galaxy. (a) Initial distribution of 500 stars in the $x$-$y$ space, (b) Final distribution after $1000/2\pi$ periods, (c) Plot of galaxy emittance $\sqrt{[(x/A)^2+(\dot{x}/\omega_0 A)^2]/2}$ as a function of the step number $t/\Delta t$, showing no sign of instability.

We obtain a growth time of $3\times10^9$ years. The galaxy cluster dimension should be smaller than $3\times10^7$ light-years. These values do not seem to be unreasonable.

We give a few further discussions below [6]:

- The case studied so far is that of a galaxy initially with infinite size. One direction of extension is to consider finite galaxies. A 1-D nonrotating galaxy model has been implemented for this purpose. In this model, each star is an infinite plane and is allowed to make finite motion only in the $z$-dimension. An unperturbed distribution with finite temperature to balance out the gravitational pull is found. The temperature turns out to be sufficient to provide stability to the galaxy by the Landau damping mechanism. A computer code was written to simulate the motion of stars in this galaxy. The result is shown in Fig.6.

- We have also implemented a 2-D rotating model. In this model, each star is a line mass infinitely long in $z$-dimension and free to move in the $x$- and $y$-dimensions. An unperturbed distribution is found when the rotating centrifugal force exactly balances the gravitational pull. This rotation requires stars to have different initial velocities. It turns out that the corresponding velocity spread is sufficient to stabilize the galaxy. A simulation of this galaxy is shown in Fig.7.

- Application can be extended to two colliding galaxies, drawing analogy to the two stream instabilities in accelerator physics. A small accidental ripple in the density distribution in one galaxy gets imprinted onto the oncoming galaxy; the perturbation on the second galaxy then enhances the initial ripple on the first galaxy by gravitational interaction, leading to instability.

- Still further extensions might take into account the special relativity and general relativity to replace Newtonian gravity. The special theory of relativity will circumvent the action-at-a-distance problem. To include general relativity, the Euclidean space-time metrics will become dependent upon $\rho$, and the problem becomes nonlinear.

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[1] See, for example, Alexander Wu Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, John Wiley & Sons, New York, 1993, and references quoted therein.

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