LINEAR OPTICS OF A SOLENOID WITH OFF-AXIS ORBIT*

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Abstract

In this paper the 6x6 linear matrix for a beam transport through a magnetic solenoid with off-axis reference orbit is derived using the hard-edge fringe field model. As an example, the result of analytical and numerical study of a bunch-compressor consisting only of solenoids and quadrupole lenses is presented.

INTRODUCTION

Magnetic solenoids have been widely used for focusing low energy beams (below 10 MeV for electrons, for example). Due to the rotational symmetry, a solenoid minimizes the number of aberrations, similar to the electrostatic round lenses. Yet, electrostatic lenses are effective in focusing only for very low energy particles. For electrons, the upper limit in energy is around 100 keV.

Due to the rotational symmetry, a magnetic solenoid with the beam on axis doesn't generate linear dispersion, nor does it cause linear dispersion in time-of-flight for relativistic particles. When the beam is off axis, the rotational symmetry is broken and the reference orbit is no longer a straight line. As a result, the off-momentum particle goes through a different path, generating linear dispersion in both transverse and longitudinal coordinates. The purpose of this paper is to derive the analytical formula for the 6x6 linear matrix for a such case.

THE OFF-AXIS ORBIT

For on-axis orbit, the traditional way of solving the equations of motion of a solenoid is using polar coordinates. When the orbit is off axis, the rotational symmetry is broken and the polar coordinates are not the easiest to use. As a result, we adopt the Cartesian coordinates. In terms of the field distribution, we choose the simplest one without losing the essence of the physics, which is the hard edge model. The on-axis field is:

$$B_z(x,y,z)=B_z(0,0,z)=B_0\theta(z)\theta(L-z),$$

where B_0 is the peak field on axis, L is the length and $\theta(z)$ is the step function defined as:

$$\theta(z)=\begin{cases}0, z<0\\ 1, z\ge 0\end{cases}.$$

To the lowest order, the off-axis field is [1]:

$$B_x(x,y,z)=-\frac{1}{2}B_z'(z)x=-\frac{1}{2}B_0[\delta(z)-\delta(L-z)]x,$$

$$B_y(x,y,z)=-\frac{1}{2}B_z'(z)y=-\frac{1}{2}B_0[\delta(z)-\delta(L-z)]y,$$

where $\delta(z)$ is the delta function.

Clearly the solenoid can be divided into three regions, i.e., the entrance, the body and the exit and the equations of motion can be solved analytically in each region. Let's consider the case that at the entrance of the solenoid $z=0^-$, the coordinates are:

$$\{x(0^-),a(0^-),y(0^-),b(0^-),\delta\}=\{x_0,a_0,0,0,0\},$$

where $a=p_x/p_0$, $b=p_y/p_0$ and $\delta=\Delta p/p_0$. Furthermore, the fifth variable is defined as

$$\Delta l=-v_0(t-t_0),$$

and it ensures that the new coordinate system is symplectic. Here v_0 is the electron velocity and t is the time. The equations of motion are:

$$\frac{dx}{dz}=\frac{a}{\sqrt{(1+\delta)^2-a^2-b^2}},$$

$$\frac{da}{dz}=-\frac{e}{p_0}\left(\frac{b}{\sqrt{(1+\delta)^2-a^2-b^2}}B_z-B_y\right),$$

$$\frac{dy}{dz}=\frac{b}{\sqrt{(1+\delta)^2-a^2-b^2}},$$

$$\frac{db}{dz}=-\frac{e}{p_0}\left(-\frac{a}{\sqrt{(1+\delta)^2-a^2-b^2}}B_z+B_x\right).$$

The equations of motion can be solved near the entrance by plugging in the field distribution and integrating from $z=0^-$ to $z=0^+$. The result is:

$$\{x(0^+),a(0^+),y(0^+),b(0^+)\}=\{x_0,a_0,0,\omega_0x_0\},$$

where $\omega_0=\dfrac{eB_0}{2p_0}$ is the Larmor frequency in z [2].

The motion in the body is simply a helix, which can be solved using geometry. For given B_0 and L, the procession angle is $2\theta_0$, where $\theta_0=\omega_0L$ is the Larmor angle. The radius of the helix is:

$$r_0=\frac{|p_\perp(0^+)|}{eB_0}=\frac{1}{2}\sqrt{\left(\frac{a_0}{\omega_0}\right)^2+x_0^2}$$

and the center is at

$$X_0=x_0-r_0\cdot\frac{\omega_0x}{\sqrt{a_0^2+\omega_0x_0^2}}=\frac{1}{2}x_0,$$

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$$Y_0 = r_0 \cdot \frac{a_0}{\sqrt{a_0^2 + \omega_0 x_0^2}} = \frac{a_0}{2\omega_0}.$$

Hence the position and momentum at the exit edge are

$$x\left(L+0^-\right) = X_0 + r_0 \cos(2\theta_0 - \theta_{c0})$$
$$= \left(x_0 \cos\theta_0\right)\cos\theta_0 + \left(\frac{a_0}{\omega_0}\sin\theta_0\right)\cos\theta_0,$$
$$y\left(L+0^-\right) = Y_0 + r_0 \sin(2\theta_0 - \theta_{c0})$$
$$= \left(x_0 \cos\theta_0\right)\sin\theta_0 + \left(\frac{a_0}{\omega_0}\sin\theta_0\right)\sin\theta_0,$$
$$a\left(L+0^-\right) = -\frac{p_\perp\left(0^+\right)}{p_0}\sin(2\theta_0 - \theta_{c0})$$
$$= -\omega_0 x_0 \sin(2\theta_0) + a_0 \cos(2\theta_0),$$
$$b\left(L+0^-\right) = \frac{p_\perp\left(0^+\right)}{p_0}\cos(2\theta_0 - \theta_{c0})$$
$$= \omega_0 x_0 \cos(2\theta_0) + a_0 \sin(2\theta_0),$$

where θ_{c0} is defined as

$$\theta_{c0} = \arctan\left(\frac{a_0}{\omega_0 x_0}\right).$$

The position and momentum after the exit edge are:

$$x\left(L+0^+\right) = x\left(L+0^-\right)$$
$$= \left(x_0 \cos\theta_0\right)\cos\theta_0 + \left(\frac{a_0}{\omega_0}\sin\theta_0\right)\cos\theta_0,$$
$$y\left(L+0^+\right) = y\left(L+0^-\right)$$
$$= \left(x_0 \cos\theta_0\right)\sin\theta_0 + \left(\frac{a_0}{\omega_0}\sin\theta_0\right)\sin\theta_0,$$
$$a\left(L+0^+\right) = a\left(L+0^-\right) + \omega_0 y\left(L+0^-\right)$$
$$= \left(-\omega_0 x_0 \sin\theta_0 + a_0 \cos\theta_0\right)\cos\theta_0,$$
$$b\left(L+0^-\right) = b\left(L+0^-\right) - \omega_0 x\left(L+0^-\right)$$
$$= \left(-\omega_0 x_0 \sin\theta_0 + a_0 \cos\theta_0\right)\sin\theta_0.$$

It is worth noting that

$$\frac{b\left(L+0^+\right)}{a\left(L+0^+\right)} = \tan\theta_0$$

which means that the angular momentum acquired at the entrance is removed at the exit.

THE 6x6 MATRIX

Based on the off-axis orbit, we can derive the linear matrix by solving the equations of motion with the initial position and momentum

$$\left\{x\left(0^-\right), a\left(0^-\right), y\left(0^-\right), b\left(0^-\right), \delta\right\}$$
$$= \left\{x_0 + \Delta x, a_0 + \Delta a, \Delta y, \Delta b, \delta\right\},$$

and keeping only the linear terms. After straightforward yet somewhat lengthy and tedious algebraic manipulation, we obtain:

$$M = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & 0 & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & 0 & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & 0 & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & 0 & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$R_{11} = \cos^2\theta_0 - \frac{\omega_0^2 x_0^2 \sin 2\theta_0 - \omega_0 x_0 a_0 \cos 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{12} = \frac{\sin 2\theta_0}{2\omega_0} - \frac{\omega_0 x_0 a_0 \sin 2\theta_0 - a_0^2 \cos 2\theta_0}{\omega_0\left(1 - a_0^2 - \omega_0^2 x_0^2\right)}\theta_0,$$

$$R_{13} = -\frac{\sin 2\theta_0}{2} + \frac{\omega_0 x_0 a_0 \sin 2\theta_0 - a_0^2 \cos 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{14} = -\frac{\sin^2\theta_0}{\omega_0} - \frac{\omega_0 x_0^2 \sin 2\theta_0 - x_0 a_0 \cos 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{16} = \frac{\omega_0 x_0 \sin 2\theta_0 - a_0 \cos 2\theta_0}{\omega_0\left(1 - a_0^2 - \omega_0^2 x_0^2\right)}\theta_0,$$

$$R_{21} = -\omega_0\left[\frac{\sin 2\theta_0}{2} + \frac{\omega_0^2 x_0^2 \cos 2\theta_0 + \omega_0 x_0 a_0 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0\right],$$

$$R_{22} = \cos^2\theta_0 - \frac{\omega_0 x_0 a_0 \cos 2\theta_0 + a_0^2 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{23} = \omega_0\left[\sin^2\theta_0 + \frac{\omega_0 x_0 a_0 \cos 2\theta_0 + a_0^2 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0\right],$$

$$R_{24} = -\frac{\sin 2\theta_0}{2} - \frac{\omega_0^2 x_0^2 \cos 2\theta_0 + \omega_0 x_0 a_0 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{26} = \frac{\omega_0 x_0 \cos 2\theta_0 + a_0 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{31} = \frac{\sin 2\theta_0}{2} + \frac{\omega_0^2 x_0^2 \cos 2\theta_0 + \omega_0 x_0 a_0 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{32} = \frac{1}{\omega_0}\left[\sin^2\theta_0 + \frac{\omega_0 x_0 a_0 \cos 2\theta_0 + a_0^2 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0\right],$$

$$R_{33} = \cos^2\theta_0 - \frac{\omega_0 x_0 a_0 \cos 2\theta_0 + a_0^2 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{34} = \frac{1}{\omega_0}\left[\frac{\sin 2\theta_0}{2} + \frac{\omega_0^2 x_0^2 \cos 2\theta_0 + \omega_0 x_0 a_0 \sin 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0\right],$$

$$R_{36} = \frac{\omega_0 x_0 \cos 2\theta_0 + a_0 \sin 2\theta_0}{\omega_0\left(1 - a_0^2 - \omega_0^2 x_0^2\right)}\theta_0,$$

$$R_{41} = -\omega_0\left[\sin^2\theta_0 + \frac{\omega_0^2 x_0^2 \sin 2\theta_0 - \omega_0 x_0 a_0 \cos 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0\right],$$

$$R_{42} = \frac{\sin 2\theta_0}{2} - \frac{\omega_0 x_0 a_0 \sin 2\theta_0 - a_0^2 \cos 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{43} = -\omega_0\left[\frac{\sin 2\theta_0}{2} - \frac{\omega_0 x_0 a_0 \sin 2\theta_0 - a_0^2 \cos 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0\right],$$

$$R_{44} = \cos^2\theta_0 - \frac{\omega_0^2 x_0^2 \sin 2\theta_0 - \omega_0 x_0 a_0 \cos 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{46} = \frac{\omega_0 x_0 \sin 2\theta_0 - a_0 \cos 2\theta_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{51} = -\frac{\omega_0 x_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0$$

$$R_{52} = -\frac{a_0}{\omega_0\left(1 - a_0^2 - \omega_0^2 x_0^2\right)}\theta_0,$$

$$R_{53} = \frac{a_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{54} = -\frac{x_0}{1 - a_0^2 - \omega_0^2 x_0^2}\theta_0,$$

$$R_{56} = \frac{1}{\omega_0}\left(\beta_0^2 - \frac{1}{1 - a_0^2 - \omega_0^2 x_0^2}\right)\theta_0.$$

Note that $\beta_0 = v_0 / c$. Using Mathematica [3], we have been able to verify that the matrix is symplectic.

Since the linear matrix is coupled, such a device tends to complicate matters in general. As for the case of on-axis orbit, a pair of identical solenoids with equal but opposite field produces an uncoupled linear matrix. One special case of such a pair that may be useful is the one where $a_0 = 0$ and $\theta_0 = \pi / 2$. The linear matrix of the solenoid pair for this case is:

$$M^T = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ R_{21}^T & -1 & 0 & 0 & 0 & R_{26}^T \\ 0 & 0 & -1 & R_{34}^T & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ R_{51}^T & 0 & 0 & 0 & 1 & R_{56}^T \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$R_{21}^T = \frac{\pi\omega_0^3 x_0^2}{1 - \omega_0^2 x_0^2}, R_{26}^T = -\frac{\pi\omega_0 x_0}{1 - \omega_0^2 x_0^2},$$

$$R_{34}^T = -\frac{\pi\omega_0 x_0^2}{1 - \omega_0^2 x_0^2}, R_{51}^T = -\frac{\pi\omega_0 x_0}{1 - \omega_0^2 x_0^2},$$

$$R_{56} = \frac{\pi}{\omega_0}\left(\beta_0^2 - \frac{1}{1 - \omega_0^2 x_0^2}\right).$$

A BUNCH COMPRESSOR

At some point we were curious if one can build a bunch compressor using only a set of off-axis solenoids and quadrupole lenses. The motivation was a significant time-off-flight delay that one can possibly obtain and seemingly easy adjustment for R_{56} that allows even changing the sign of R_{56}. Thus, we studied the simplest possible scheme with the bunch compressor consisting only of three pairs of solenoids and two quadrupoles. The quadrupoles are placed before and after the center solenoid pair. The approximate solution was found first using Mathematica assuming hard edge fringe field and thin quadrupoles. A more realistic model was established using COSY INFINITY [4] assuming the fringe field with the shape of a hyperbolic tangent function and finite length quadrupoles. The main parameters are listed below.

Beam energy	E_{k0}	100	MeV
Solenoid field	B_0	1.266	T
Larmor angle	θ_0	75.73	deg.
Solenoid length	L	0.7	m
Solenoid radius	R	3	cm
Distance	L_D	4.6	m
Quad gradient	kq	0.1066	1/m
Linear compression	R_{56}	5.20	cm
Quadratic compression	T_{566}	-225	cm
Total length	L_{tot}	14.3	m

Note that R is the inner radius of the solenoids and L_D is the distance between the center and the outer solenoid pairs. The fact that θ_0 is not 90 degrees is due to the finite width of the fringe region (the solenoids within one pair are 30 cm apart). The analytical solution is 1.5 T and 90 degrees. The large T_{566} (~ 40 times R_{56}) compared to the conventional bend magnet compressor (1.5 times R_{56}) makes it impractical. But if very large T_{566} is needed in some application, the magnetic solenoid bunch compressor could be a viable solution.

REFERENCES

[1] P. W. Hawkes and E. Casper, "Principles of Electron Optics" (Academic Press, London, 1996), p. 87.
[2] M. Reiser, "Theory and Design of Charged Particle Beams", (John Wiley&Sons, New York, 1995), p. 35.
[3] Wolfram Research, Mathematica, http://www.wolfram.com.
[4] M.Berz *et. al.*, COSY INFINITY, http://www.bt.pa.msu.edu/index_cosy.htm