

# GENERALIZED KAPCHINSKIJ-VLADIMIRSKIJ SOLUTION FOR WOBBLING AND TUMBLING BEAMS IN A SOLENOIDAL FOCUSING LATTICE WITH TRANSVERSE DEFLECTING PLATES \*

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## Abstract

For applications of high-intensity beams in heavy ion inertial confinement fusion and high energy density physics, solenoidal focusing lattice and transverse wobblers can be used to achieve uniform illumination of the target and for suppressing deleterious instabilities. A generalized self-consistent Kapchinskij-Vladimirskij solution of the nonlinear Vlasov-Maxwell equations is derived for high-intensity beams in a solenoidal focusing lattice with transverse wobblers. The cross-section of the beam is an ellipse with dynamical centroid, tilting angle, and transverse dimensions that are determined from 5 envelope-like equations.

## INTRODUCTION

Important application areas of high-intensity ion beams include heavy ion inertial fusion and high energy density physics. To deliver enough kinetic energy to the target for the purpose of reaching high-gain conditions for inertial confinement fusion or creating matter in the high energy density regime, the intensity of the driver beams is required to approach the space-charge limit of accelerators and beam transport systems [1].

Another requirement is that the high-intensity beams need to generate a smooth, uniform illumination of the target to suppress deleterious instabilities, such as the Rayleigh-Taylor instability, normally associated with acceleration and non-uniformity of the target [2]. Currently, there are two complementary techniques envisioned to achieve this goal. First, we can induce beam spinning dynamics in the transverse plane by using a solenoidal lattice, which can also be used to focus the beam onto the target. The spinning of the beam will smooth out the intensity non-uniformity over the beam cross-section. If the target size is larger than the beam focal spot size, we can also generate a wobbling dynamics of the beam centroid to scan the target to provide uniform illumination. The wobbling dynamics of the beam centroid can be generated by transverse kicks of the beam particles by a series of transverse deflecting plates driven by rf potential. Such devices are called “wobblers”. The rf potential on the wobbler plates depends on time so that different slices of the beam are deflected differently and delivered to different locations on the target to achieve the smoothing effect. (see Fig. 1).

The purpose of this paper is to provide a self-consistent kinetic description of a high-intensity beam in a solenoidal lattice with wobblers. The dynamics of high-intensity beams is described by the nonlinear Vlasov-Maxwell equa-

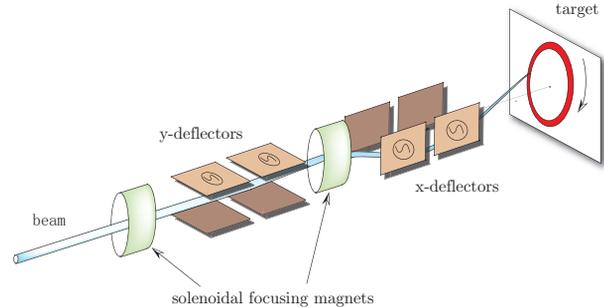


Figure 1: Solenoidal lattice with wobblers.

tions [3]. In this paper, we derive a generalized self-consistent Kapchinskij-Vladimirskij (KV) solution for the system. Recall that the classical KV solution [4] is the only known exact solution of the nonlinear Vlasov-Maxwell equations in an alternating-gradient quadrupole focusing lattice. The classical KV solution is a delta-function of the weighted sum of the two Courant-Snyder invariants in the two transverse directions. The cross-section of the beam is an ellipse with dynamical transverse dimensions given by the envelope functions, which are determined from the corresponding envelope equations.

For the generalized KV solution in a solenoidal lattice with wobblers derived in this paper, the beam cross-section is also an ellipse. But the centroid  $(\mu, \nu)$ , tilting angle  $\theta$ , and transverse dimensions  $(A, B)$  of the ellipse are all functions of time, and they are determined by 5 envelope-like equations (see Fig. 2).

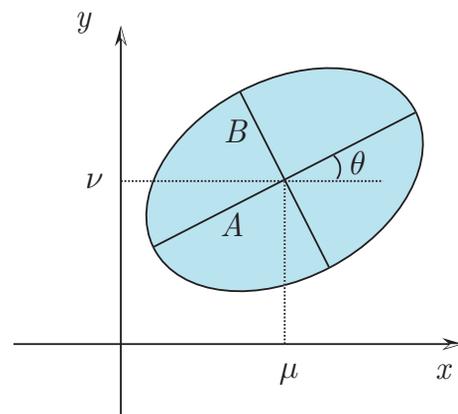


Figure 2: Wobbling and tumbling beam with centroid  $(\mu, \nu)$ , tilting angle  $\theta$ , and transverse dimensions  $(A, B)$ .

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## GENERALIZED KV SOLUTION

In a solenoidal focusing lattice with wobblers, a particle's dynamics in the laboratory-frame coordinate system  $(x, y)$  is determined from [3]

$$x'' = 2\Omega y' + \Omega' y - \frac{\partial\psi}{\partial x} - F_x(s), \quad (1)$$

$$y'' = -2\Omega x' - \Omega' x - \frac{\partial\psi}{\partial y} - F_y(s), \quad (2)$$

where  $\psi = e\phi/\gamma^3 m\beta^2 c^2$  is the normalized space-charge potential,  $\Omega(s) = eB(s)/2\gamma m\beta c^2$  is the normalized Larmor frequency of the solenoidal lattice, and  $F_x(s)$  and  $F_y(s)$  are the transverse deflection force components due to the wobblers. The nonlinear Vlasov-Maxwell equations for the beam distribution function  $f(s, x, y, v_x, v_y)$  and space-charge potential  $\psi$  are [3]

$$\frac{\partial f}{\partial s} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + \left( 2\Omega y' + \Omega' y - \frac{\partial\psi}{\partial x} - F_x \right) \frac{\partial f}{\partial v_x} + \left( -2\Omega x' - \Omega' x - \frac{\partial\psi}{\partial y} - F_y \right) \frac{\partial f}{\partial v_y} = 0, \quad (3)$$

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = -\frac{2\pi K_b}{N_b} \int f dv_x dv_y, \quad (4)$$

where  $N_b = \int f dv_x dv_y dx dy$  is the line density of the beam particles, and  $K_b = 2N_b e/\gamma^3 m\beta^2 c^2$  is the self-field permeance. We assume that there is no coupling between different slices of the beam, and that the nonlinear Vlasov-Maxwell equations describe the transverse dynamics of each slice of the beam. It is further assumed that the conducting wall is far away from the beam.

Our objective is to find a self-consistent solution of Eqs. (3) and (4). We start from the dynamics of the beam centroid. We define the beam centroid  $(\mu, \nu)$  by the solutions of the centroid equations

$$\mu'' = 2\Omega \nu' + \Omega' \nu - F_x(s), \quad (5)$$

$$\nu'' = -2\Omega \mu' - \Omega' \mu - F_y(s). \quad (6)$$

Note that the beam centroid is subject to the focusing lattice and the wobbler forces, but not to the self-generated space-charge force. Subtracting Eqs. (5) and (6) from Eqs. (1) and (2), we obtain

$$\tilde{x}'' = 2\Omega \tilde{y}' + \Omega' \tilde{y} - \frac{\partial\psi}{\partial \tilde{x}}, \quad (7)$$

$$\tilde{y}'' = -2\Omega \tilde{x}' - \Omega' \tilde{x} - \frac{\partial\psi}{\partial \tilde{y}}, \quad (8)$$

$$\tilde{x} \equiv x - \mu, \quad \tilde{y} \equiv y - \nu. \quad (9)$$

Here  $\tilde{x}$  and  $\tilde{y}$  are the particle displacements relative to the beam centroid (see Fig. 2), and we have assumed that  $\psi$  depends on  $(x, y)$  only through  $(\tilde{x}, \tilde{y})$ , i.e.,  $\psi = \psi(\tilde{x}, \tilde{y})$ . We will confirm, for the solution found, that this assumption is indeed satisfied. Now, we transform the dynamical

equations to the Larmor frame defined by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = R(\theta) \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \quad (10)$$

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta = -\int_{s_0}^s \Omega ds. \quad (11)$$

A straightforward calculation shows that in the Larmor frame, defined by the  $(X, Y)$  coordinates, the dynamical equations of motion are given by

$$X'' = -\Omega^2 X - \frac{\partial\psi}{\partial X}, \quad (12)$$

$$Y'' = -\Omega^2 Y - \frac{\partial\psi}{\partial Y}, \quad (13)$$

and the Vlasov-Maxwell's equations can be expressed as

$$\frac{\partial f}{\partial s} + V_X \frac{\partial f}{\partial x} + V_Y \frac{\partial f}{\partial y} - \left( \Omega^2 X + \frac{\partial\psi}{\partial X} \right) \frac{\partial f}{\partial V_X} - \left( \Omega^2 Y + \frac{\partial\psi}{\partial Y} \right) \frac{\partial f}{\partial V_Y} = 0, \quad (14)$$

$$\left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) \psi = -\frac{2\pi K_b}{N_b} \int f dV_X dV_Y. \quad (15)$$

Following the spirit of the KV formulation, we search for a solution to Eqs. (14) and (15) with

$$\psi(s, X, Y) = \frac{-K_b}{A+B} \left[ \frac{X^2}{A} + \frac{Y^2}{B} \right], \quad (16)$$

for  $\frac{X^2}{A^2} + \frac{Y^2}{B^2} \leq 1$ .

Here  $A$  and  $B$  are time-dependent envelope functions in the transverse  $(X, Y)$  plane. For this form of  $\psi$ , Maxwell's equation (15) becomes

$$\int f dV_X dV_Y = \frac{N_b}{\pi AB}, \quad \text{for } \frac{X^2}{A^2} + \frac{Y^2}{B^2} \leq 1. \quad (17)$$

We now select  $A$ ,  $B$ , and  $f$  such that Eqs. (14) and (17) are satisfied. First, we note for  $\psi$  given by Eq. (16), the space-charge force is linear, and the  $(X, Y)$  dynamics are decoupled, i.e.,

$$X'' = -\Omega^2 X - \frac{2K_b X}{(A+B)A}, \quad (18)$$

$$Y'' = -\Omega^2 Y - \frac{2K_b Y}{(A+B)B}. \quad (19)$$

Equations (18) and (19) are linear equations for  $X$  and  $Y$  with time-dependent focusing coefficients. Obviously, there is one Courant-Snyder invariant [5] for each transverse direction,

$$I_X = \frac{X^2}{W_X^2} + (W_X X' - X W_X')^2 = \text{const.},$$

$$I_Y = \frac{Y^2}{W_Y^2} + (W_Y Y' - Y W_Y')^2 = \text{const.},$$

where  $W_X$  and  $W_Y$  are determined through the envelope equations

$$A'' + \Omega^2 A - \frac{2K_b}{(A+B)} = \frac{\varepsilon_X^2}{A^3}, \quad (20)$$

$$B'' + \Omega^2 B - \frac{2K_b}{(A+B)} = \frac{\varepsilon_Y^2}{B^3}, \quad (21)$$

$$W_X \equiv \frac{A}{\sqrt{\varepsilon_X}}, \quad W_Y \equiv \frac{B}{\sqrt{\varepsilon_Y}}. \quad (22)$$

Here, the constants  $\varepsilon_X$  and  $\varepsilon_Y$  are the emittance in the two transverse directions. Since  $I_X$  and  $I_Y$  are constants of the motion, any function of the form  $f(I_X, I_Y)$  is a solution of the nonlinear Vlasov equation (14). To satisfy Eq. (17), we select

$$f = \frac{N_b}{AB} \delta \left( \frac{I_X}{\varepsilon_X} + \frac{I_Y}{\varepsilon_Y} - 1 \right), \quad (23)$$

It is straightforward to show that

$$\int f dV_X dV_Y = \begin{cases} \frac{N_b}{\pi AB}, & \frac{X^2}{A^2} + \frac{Y^2}{B^2} \leq 1, \\ 0, & \frac{X^2}{A^2} + \frac{Y^2}{B^2} > 1. \end{cases} \quad (24)$$

To summarize, the nonlinear Vlasov-Maxwell equations (3) and (4) in a solenoidal lattice with wobblers admit a solution in the form of Eqs. (23) and (16), and the solution is specified by solving the envelope-like equations (5), (6), (11), (20), and (21). In general, the envelope-like equations can be viewed as a set of ordinary differential equations in time, to which the nonlinear Vlasov-Maxwell equations in phase space reduce for the class of solutions constructed here.

One interesting feature of the solution is that the beam needs not have a circular cross-section, even though it does include circular cross-section solutions as a special case. Previous studies for a solenoidal lattice normally consider only beams with circular cross-section. In particular, we can use the solution derived here to study how the beam can be perturbed away from an equilibrium with circular cross-section.

To illustrate this point, let's consider the case where  $\Omega = \text{const.}$  and  $\varepsilon_X = \varepsilon_Y = \varepsilon$ . The equilibrium solution of Eqs. (20) and (21) is

$$A = B = R, \quad (25)$$

$$\Omega^2 R - \frac{K_b}{R^2} = \frac{\varepsilon}{R^3}. \quad (26)$$

Consider a small perturbation with

$$A = R + \delta A, \quad (27)$$

$$B = R + \delta B. \quad (28)$$

Linearizing Eqs. (20) and (21) in terms of  $\delta A$  and  $\delta B$ , and assuming  $(\delta A, \delta B) \sim \exp(i\omega t)$ , we obtain the matrix

equation

$$\begin{pmatrix} -\omega^2 + \Omega^2 + \frac{K_b}{2R^2} + \frac{3\varepsilon}{R^4} & -\omega^2 + \Omega^2 + \frac{K_b}{2R^2} + \frac{3\varepsilon}{R^4} \\ \frac{K_b}{2R^2} & \frac{K_b}{2R^2} \end{pmatrix} \cdot \begin{pmatrix} \delta A \\ \delta B \end{pmatrix} = 0, \quad (29)$$

which gives two eigenvalues

$$\omega_1 = \sqrt{4\Omega^2 - \frac{2K_b}{R^2}}, \quad (30)$$

$$\omega_2 = \sqrt{4\Omega^2 - \frac{3K_b}{R^2}}. \quad (31)$$

For the first eigenvalue,  $\omega = \omega_1$ , the polarization of the mode is  $\delta A = \delta B$ , which implies that the cross-section of the beam is a pulsating circle. In the space-charge limit, the mode frequency is  $\omega = \omega_1 = \sqrt{2}\Omega$ . For the second eigenvalue,  $\omega = \omega_2$ , the polarization of the mode is  $\delta A = -\delta B$ . In this case, the cross-section of the beam is a pulsating ellipse, deviating from a circular shape. In the space-charge limit, the frequency of this mode approaches  $\omega = \omega_1 = \Omega$ .

## CONCLUSION AND FUTURE WORK

For high-intensity beams in a solenoidal lattice with wobblers described by the nonlinear the Vlasov-Maxwell equations, a class of generalized KV solutions is found. The self-consistent solutions are specified by 5 envelope equations for the beam centroid (wobbling dynamics), the tilting angle (tumbling dynamics), and the transverse dimensions in the wobbling and tumbling frame. Solenoidal lattice is one of the two techniques that can induce coupling between the transverse dynamics to achieve the smoothing effect in the transverse dimension. The other technique is to use a skew-quadrupole lattice. A similar class of solutions of the nonlinear Vlasov-Maxwell equations for high-intensity beams in a skew-quadrupole lattice with wobblers has also been discovered [6], and will be reported in future publications.

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