

# RELAXATION OF INTENSE INHOMOGENEOUS MISMATCHED CHARGED BEAMS\*

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## Abstract

This work analyzes the dynamics of inhomogeneous, magnetically focused high intensity beams of charged particles. Initial inhomogeneities lead to density waves propagating transversely in the beam core, and the presence of transverse waves eventually results in particle scattering. Particle scattering off waves in the beam core ultimately generates a halo of particles with concomitant emittance growth. Emittance growth indicates a beam relaxing to its final stationary state, and the purpose of the present paper is to describe halo and emittance in terms of test particles moving under the action of the inhomogeneous beam. To this end an average Lagrangian approach for the beam is developed. This approach, aided by the use of conserved quantities, produces results in nice agreement with those obtained with full  $N$ -particle numerical simulations.

## INTRODUCTION

Magnetically focused beams of charged particles can relax from non-stationary to stationary flows with associated emittance growth and concomitant halo formation [1]. Gluckstern [2] showed that initial envelope oscillations of mismatched homogeneous beams induce formation of large scale resonant islands beyond the beam border [3, 4]: beam particles are captured by the resonant islands resulting in emittance growth and relaxation. A closely related question concerns the mechanism of beam relaxation and the associated emittance growth when the beam is not homogeneous, as frequently happens in beam transport channels [1, 5]. On general grounds of energy conservation one again concludes that beam relaxation takes place when the coherent fluctuations of beam inhomogeneities are converted into microscopic kinetic energy, as shall be detailed along the paper.

Recent investigation of inhomogeneous beams shows that relaxation comes about as a consequence of particle scattering off density waves in the beam [6]. Scattering particles initially move in-phase with the macroscopic density fluctuations, drawing their energy from the propagating wave fronts and converting it into microscopic kinetic energy. For ultra cold, or crystalline beams, the process amounts to the mechanism of pure wave breaking, where

particles are firstly coherently accelerated to the velocity of the waves and then abruptly ejected from high density peaks. At the moment of ejection, spatial dependence of oscillatory frequency, a needed feature for wave breaking, has already turned the core into a highly incoherent state [7, 8].

In the present paper we focus attention on the case of space charge dominated but warmer beams. Under these conditions, resonant particles are already present at initial times due to thermal spread, and the entire relaxation process is smoother. In contrast to the crystalline case, here particles gain energy while the core still displays coherence. In any case, ejected particles form a low density halo around the beam core, which ultimately increases beam emittance and relaxes the dynamics. Due to its low density the ejected population can be very accurately described as a set of test particles. It is thus of importance to describe the motion of test particles as they are driven into the halo by core fluctuations. The core itself generically behaves as an oscillatory drive, the details of which depend on the particular instance investigated. While in homogeneous cases the core is occasionally modeled as a breathing flat top charge distribution, corresponding models for oscillating inhomogeneous beams are less frequent. We perform our investigation with aid of average Lagrangean techniques.

## THE MODEL AND ANALYSIS

Considering full azimuthal symmetry, one can use Gauss's law in order to write the governing equation for any particle in the beam [7, 9, 10, 11],

$$r'' = -\kappa r + \frac{Q(r)}{r}, \quad (1)$$

primes indicating derivative with respect to the longitudinal  $z$  coordinate which for convenience we shall also refer to as "time".  $\kappa$  is the focusing factor  $\kappa \sim B^2$ , where  $B$  is the axial, constant, focusing magnetic field and  $Q(r)$  is the beam charge (actually, the beam "perveance") up to radial position  $r$ .

### *Density Oscillations of the Core*

We recall that we are interested in slightly thermal beams where particles begin to move away from the core while it still oscillates coherently.

Since the core is assumed to be a fluid with discrete random motion, the amount of charge that a core particle sees at any time equals the charge initially seen at  $z = 0$ . In

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other words, if a core particle evolves from  $r_0$  at  $z = 0$  to a new position  $r$  at time  $z$ , we consider  $Q(r, z) = Q(r_0)$ .  $r_0$  is in fact the Lagrangian coordinate of the core particle [12], which means that the solution to Eq. (1) can be written parametrically in terms of  $r_0$  in the convenient form  $r = r(r_0, z)$ . Once again we emphasize that the amount of charge  $Q(r)$  seen by the fluid element inside the region  $0 < r \leq r(z)$  remains unaltered at  $Q(r_0)$ , independently of time  $z$ . This is of fundamental importance since from Gauss's law this is the charge that exerts the force on the fluid element.

Expression (1), adapted for core particles according to the preceding comments, can be readily obtained from the single-particle Lagrangian  $\mathcal{L}$

$$\mathcal{L}(r, r') = \frac{r'^2}{2} - \kappa \frac{r^2}{2} + Q(r_0) \ln(r) \quad (2)$$

with help of Euler-Lagrange equations. In our system one has a multitude of  $N_t$  particles, as mentioned, and the full transverse Lagrangian takes the form

$$L = \int \mathcal{L}(r, r') n(r_0) d^2 r_0, \quad (3)$$

where what one is doing is to multiply the single-particle Lagrangian at coordinate  $r$  by the number of particles evolving from  $r(z = 0) = r_0$  to  $r(z) = r$ ,  $n(r_0) d^2 r_0$ , and integrating over all possible initial conditions. Recalling that since we are dealing with core particles, the amount of charge seen by a particle at any time  $z$  equals that at  $z = 0$ . This explains the presence of the term  $Q(r_0)$  - computed at  $r_0$  - in Eq. (2).

At the present point we would like to invoke average Lagrangian techniques [13]. The purpose here would be to produce an average formalism that could provide an easy way into obtaining the nonlinear frequency and the amplitude of the dominant oscillatory mode of the core. With this approximate core dynamics serving as a drive for test particles, we shall finally attempt to investigate halo formation and the corresponding basic features of the relaxed state.

The general idea of the average Lagrangian is to suppose a trial shape for the density  $n = n(r, \chi(z))$ , where  $z$  dependence comes through an amplitude factor  $\chi = \chi(z)$  to be determined. Then one proceeds to integrate Eq. (3) over  $r_0$  and apply Euler-Lagrange method of stationary action to obtain a governing equation for the density fluctuation amplitude  $\chi(z)$ . This will complete the description for the core dynamics.

As initial condition we impose a parabolic type perturbation of the form  $n(r \leq r_c) = \rho_h [1 + \chi(2r^2/r_c^2 - 1)]$  (with  $n(r > r_c) = 0$ ), where  $\rho_h$  is the average beam density and  $r_c$  is the beam core radius. Then we assume that as the density wave evolves in the core it can be represented in the form

$$n(r, \chi(z)) = \rho_h \left[ 1 + \chi(z) \left( 2 \frac{r^2}{r_c^2} - 1 \right) \right], \quad (4)$$

## Beam Dynamics and Electromagnetic Fields

where now we let the amplitude to become a function of time  $z$ . The *ansatz* represents a compressive-rarefactive wave and gives an accurate account of the density fluctuations for short times following the initial state. Since the quickly evaporated particles are the ones that absorb maximum energy and define the halo boundaries in phase space, usage of expression (4) is justified. We also choose  $r_c = K/\kappa$  so particles at beam border are at equilibrium.

With help of the *ansatz* and all steps detailed above, we can fully integrate Eq. (3) with respect to  $r$  and make use of Euler-Lagrange equations applied to the remaining function  $\chi = \chi(z)$  to obtain a closed expression for the amplitude

$$\chi''(z) = F(\chi(z), \chi'(z)). \quad (5)$$

where  $F$  is an involved function that can be nevertheless written in terms of  $\chi$  and  $\chi'$ .

## Test Particle Orbits

As for test particles, Eq. (1) is applied with the restriction that  $Q(r)$  contains only the core charge. This is equivalent to our assumptions on the diluteness of the halo, whereby the core drives test particles but is not acted upon by the latter.

As they interact with the beam, test particles feel the space charge action of the core up to their current position  $r = r(z)$ . In particular, test particles outside the beam see constant charge. Therefore the governing equation for those test particles can be written as

$$r'' = -\kappa r + \begin{cases} K \frac{r(r_c^2 + (r^2 - r_c^2)\chi(z))}{r_c^4} & \text{if } r \leq r_c \\ K/r & \text{if } r > r_c \end{cases}. \quad (6)$$

Equations (5) and (6) shall be solved simultaneously to obtain the dynamics of test particles.

## ESTIMATES VERSUS FULL SIMULATIONS

### Estimates Based on Conserved Quantities and Test Particle Dynamics.

The  $\chi$  dynamics, coming from a time independent 1D Lagrangian, is completely integrable and periodic. Test particle motion can be thus represented in terms of a convenient Poincaré plot, where we record the pair of values  $r(z)$  and  $r'(z)$  each time  $\chi(z)$  cycles one period.

One can also rescale radiuses, perveance and focusing factor so as to work with  $K = \kappa = r_c = 1$  [14]. If in addition one selects  $\chi(z = 0) \equiv \chi_0 = 0.4$  and  $\chi'(0) \rightarrow 0$  as typical initial conditions at beam entrance, one obtains the dynamics for the test particles as the scattered plot shown in the Fig. 1.

Except for a resonance bubble near the origin, test particles distribute evenly over a limited region of phase space delimited by the orbit of the fastest test particles at  $r = 0$ . The boundary is the thick line present in Fig. (1). One

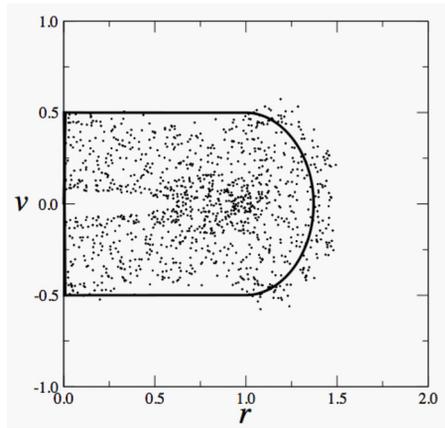


Figure 1: Test particle dynamics (scattered dots) and bounding energy curve (thick line) for  $\chi_0 = 0.4$ .

key assumption of the model is that the halo will be homogeneously distributed over the region bounded by this maximum energy curve.

Test particles calculations compare extremely well with full simulations of Fig. 2.

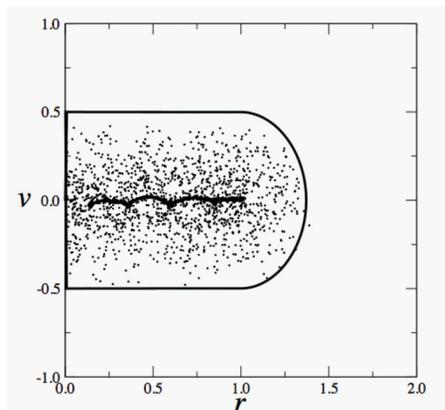


Figure 2: Full simulations displaying a snapshot of the phase space occupied by  $N_t = 10000$  macroparticles at  $z = 5000$ , for  $\chi_0 = 0.4$ . The thick line representing bounding energy is once again displayed.

What remains unknown up to the present point is the halo normalization - or the total number of halo particles - within the bounded domain. Only full knowledge of the distribution, layout and normalization, enables to calculate several quantities for the relaxed beam, including the emittance, an instrument of choice for beam diagnostics.

The number of halo particles can be obtained through a combination of energy conservation, envelope equation and Poisson equation. Details can be found in Ref. [15]. Results for the emittance  $\varepsilon$  indicating good agreement between analytical approximations and full simulations can be found in the Table.

Table 1: Comparison of full simulations and analytical estimates for asymptotic states based on test particle dynamics and conserved quantities.

$\chi_0$	$\varepsilon_{analytical}$	$\varepsilon_{simulation}$
0.2	0.053	0.051
0.4	0.106	0.099
0.6	0.152	0.140
0.8	0.211	0.175

## FINAL REMARKS

We employed average Lagrangean techniques to estimate saturation of wave breakein in intense inhomogeneous beams. Test particles are launched under the action of fields calculated with the average techniques, and estimates compare well with full particle simulations.

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