ANISOTROPIC KINETIC AND DYNAMIC PROCESSES IN EQUIPARTITIONED BEAMS*

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Abstract

The purpose of this paper is to propose one definiton of the anisotropic equipartition [1]. Anisotropic equipartition corresponds to a phase space density uniform on the surface invariant of the ξ, where ξ is the ratio of oscillations energies in the x and y directions, a version of the ergodic hypothesis where the ξ invariant play the role of the conserved energy [2]. In the state of anisotropic equipartition, the beam temperature is stationary, the entropy grows in the cascade form, there is a coupling of transversal emittance, the beam develops an elliptical shape with a increase in its size along one direction and there is halo formation along one direction preferential.

INTRODUCTION

Space-charge interactions in high-intensity linear accelerator can lead to equipartitioning of energy between the degrees of freedom. The question then is whether an anisotropic system of collisionless particles coupled by long-range space-charge forces will equipartition and, if so, how. In anisotropic beams, the emittance and/or external focusing force strength are different in the two transverse directions. Ikegami [3] deal with effects of anisotropic of beam cores on halo dynamics. Recently, other news phenomena caused by anisotropy demonstrated by Hofmann *et al.* [3].

We are working toward quantifying the relationship between the anisotropy of the beam and the equipartition. The equipartition of beam is driven for anisotropics processes. In plasmas with strongly anisotropic distribution functions, collective instabilities may develop if there is sufficient coupling between the degrees of fredom. Previous studies have mostly focused on the electrostatic Harris-type anisotropic-driven instability for beams [2]. The term equipartition broadly refers to the ergodic poperty of multidimensional Hamiltonian systems, which tend to to distribute uniformly over the phase space surface of constant energy. The conservation of energy plays the fundamental role in classical equilibrium thermodynamics. The temperature is then uniform. The term "turbulent equipartiton " was introduced by Yankov [2] in order to describe the turbulent relaxed state, in which the system assumes a uniform distribution on the surface of constant invariants respected by turbulence. Previous examples of TEP in plasmas physics were considered by Yankov [2] and Isichenko *et al.* [2].We believe that the problem of anisotropic equipartition in linac should be solved in the same spirit.

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ANISOTROPIC EQUIPARTITION

We consider an axially long unbunched beam of ions of charge q and mass m propagating with average axial velocity $\beta_b c\hat{\mathbf{e}}_z$ along an uniform linear focusing channel, self-field interactions are electrostatic. We consider the parabolic density beam ($n_b = 2N_b/\pi r_x r_y \left[1 - x^2/r_x^2 - y^2/r_x^2\right]$) where N_b is the axial line density, $r_x = \sqrt{6\langle x^2\rangle}$ and $r_y = \sqrt{6\langle y^2\rangle}$ are ellipsis semi-axes *rms*, in space-charge dominated regime [1]. The envelope of the beam is an elliptical cross-section with *rms* radii r_j (henceforth, j ranges over both x and y) that obey the *rms-KV* envelope equations [1], $r_j^{''} + \kappa_0^2 r_j - \frac{2K}{r_x+r_y} - \frac{\epsilon_j^2}{r_j^3} = 0$. Here, $K = q^2N_b/\pi^2\varepsilon_0\gamma_b^3\beta_b^2 mc^2$ is the dimensionless perveance of the beam and κ_0 is represented constant focusing force. There is a solution of the envelope equations for which $r_j(s) = r_{b0} = \left[\left(K + (K^2 + 4\kappa_0^2\eta^2)^{1/2}\right)/2\kappa_0^2\right]^{1/2}$, where $\eta = \epsilon_x/\epsilon_y$, this corresponds to the so called matched solution. ϵ_j is *rms*-emittance of the beam along the j-plane. The $\epsilon_j = \sqrt{\langle j^2\rangle\langle j'^2\rangle - \langle jj'\rangle^2}$ was calculated analytically, following a model proposed to Simeoni [1]. The emittance is given by :

$$\epsilon_x = \frac{1}{90}\sqrt{15}KA\left\{\frac{\left(\frac{r_x}{r_y}\right)^2\left[5\left(\frac{r_x}{r_y}\right)^2 + 2\frac{r_x}{r_y} + 5\right]}{\left(1 + \frac{r_x}{r_y}\right)^4}\right\}^{1/2} \tag{1}$$

the result is easily transformed to the y plane interchanging r_x and r_y. A is the mismatched amplitude of oscillatories modes beam. We introduce the following anisotropy variables: the ratio emittance η, ratio of the envelope beam $\chi = r_x/r_y$, and the mismatch factor $\nu = r_x/r_{b0} = r_y/r_{b0}$. We launch the beam with $K = 3$, $\kappa_0 = 1$, $\nu = 2.4$, $A = 2.0$, $r_{j0} = \nu r_{b0}$, $r_{x0} = r_{y0}$ and $\eta = 1$ initially.

In the proceeding [1], we showed that there is a coupling of transversal emittance, the beam develops an elliptical shape with a increase in its size along one direction and there is halo formation along one direction preferential, as consequence of resonant phase mixing [2]. For large beam size-rms mismatched and initial ratio envelopes beam $\chi = 1$, the ratio of oscillations energies in the x and y directions remaines constant, $\xi = (r_y\epsilon_x)^2/(r_x\epsilon_y)^2 = 1$, [3]. As illustrated in Fig. 1 beam fast suffers anisotropization characterized for discontinuous variations in χ and η [2]. The anisotropy leading to coupling resonance in the presence of nonlinear space-charge forces was suggested as a possible approach to the equipartitioning question [3].

Emittance is the area in phase space occupied by the par-

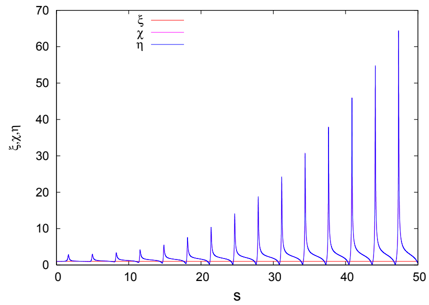

Figure 1: The ratio of oscillations energies ξ and anisotropic ratios, χ and η in beam propagation.

ticles of the beam. The maximum emittance of a beam that a system can accept is called the acceptance of the system a [3]. We define the RMS acceptance as the maximum square root of the $a_j = \sqrt{\langle j^2\rangle\langle j'^2\rangle - \langle jj'\rangle^2}$. The second moment term $\langle jj'\rangle^2$ represents a correlation between j and j' that exists when the beam envelope is converging or diverging. At a waist, this correlation is minimized and the second moment term is zero [1]. Thus, the acceptance reduces to maximum of the $\tilde{a}_j = \sqrt{\langle j^2\rangle\langle j'^2\rangle}$.With the help of the Lyapunov functions [3] we can construct an invariant beam area. A system, $\frac{dx(t)}{dt} = f(x)$, takes the form $\frac{dx(t)}{dt} = -\frac{dL(x)}{dx}$, if to an equilibrium x_0 the Lyapunov function L satisfies the conditions $L(x_0) = 0$, $\frac{dL(x)}{dt} \leq 0$ for $x \neq x_0$ and L is a smooth function of x in some neighborhood of x_0, details in [1]. A definition of the Lyapunov function of the acceptance $L(\tilde{a}_j)$ would be :

$$\begin{aligned} L &= \alpha\left(1-\sqrt{1/\xi}\right)\frac{\tilde{a}_x^2}{2r_x} - \rho\left(1+1/\eta\right)\frac{\tilde{a}_x^2}{2} \\ &- \alpha\left(\sqrt{\xi}-1\right)\frac{\tilde{a}_y^2}{2r_y} - \rho\left(1+\eta\right)\frac{\tilde{a}_y^2}{2} \end{aligned} \quad (2)$$

This function is dependent of the ratio of the average external focusing force to the space-charge force α and of the space-charge strength ρ. To $\tilde{a}_{x_0} = 0$ and $\tilde{a}_{y_0} = 0$, L satisfies the condition $L(\tilde{a}_{x_0}, \tilde{a}_{y_0}) = 0$. To the issue $\frac{dL(\tilde{a}_j)}{ds} \leq 0$ mentioned above the evolution of the $L(\tilde{a}_j)$ Lyapunov function represented by blue line (top graphic) in Fig. 3 can be regarded as a proof. As the Lyapunov functionc satisfies the conditions, we can apply equation $\frac{d\tilde{a}_j(s)}{ds} = -\frac{dL(\tilde{a}_j)}{d\tilde{a}_j}$ obtaining the following acceptance dynamics equations:

$$\frac{d\tilde{a}_x}{ds} = -\alpha\left(1-\sqrt{1/\xi}\right)\frac{\tilde{a}_x}{r_x} + \rho\left(1+1/\eta\right)\tilde{a}_x \quad (3)$$

$$\frac{d\tilde{a}_y}{ds} = \alpha\left(\sqrt{\xi}-1\right)\frac{\tilde{a}_y}{r_y} + \rho\left(1+\eta\right)\tilde{a}_y \quad (4)$$

The resulting acceptance equations contain two terms: the first term describe acceptance changes associated with transfer of energy between the two planes; the second describes acceptance changes associated with anisotropic processes.The general solutions to the equations (3) and (4) are given by:

$$\tilde{a}_x = c_1 e^{\int\left[\frac{\alpha}{r_x}\left(\sqrt{1/\xi}-1\right)+\rho(1+1/\eta)\right]ds}, \quad (5)$$

$$\tilde{a}_y = c_2 e^{\int\left[\frac{\alpha}{r_y}\left(\sqrt{\xi}-1\right)+\rho(1+\eta)\right]ds} \quad (6)$$

where c_1 and c_2 are arbitrary constants determined by initial values.The oscillations beam envelope perturb nonlinear space-charge force yields a correlation between particle position and transverse momentum. Thus, the second moment term $\langle jj'\rangle^2$ is not zero and the RMS acceptance becomes $a_j = \sqrt{\langle j^2\rangle\langle j'^2\rangle - \langle jj'\rangle^2}$. Therefore the equipartition and the variable anisotropy are given by $\xi = (r_y a_x)^2/(r_x a_y)^2$ and $\eta = a_x/a_y$, respectively. Thus the equations (3) and (4) are transformed by :

$$\frac{da_x}{ds} = \left(\frac{-\alpha}{r_x}+\rho\right)a_x + \left(\frac{\alpha}{r_y}+\rho\right)a_y, \quad (7)$$

$$\frac{da_y}{ds} = \left(\frac{\alpha}{r_x}+\rho\right)a_x + \left(\frac{-\alpha}{r_y}+\rho\right)a_y \quad (8)$$

Jameson has derived identical equations [3] to the question of equipartioning in linear accelerator. The general solutions to the equations (7) and (8) are given by [1]:

$$\begin{aligned} a_x &= -\left\{c_3\int\left[\left(e^{2\rho s+\alpha\int\left(-\frac{1}{\chi r_y}+\frac{\chi}{r_x}\right)ds}\right)(\alpha+\rho r_x)\right.\right. \\ &\left.\left.\frac{1}{r_x}\right]ds + c_4\right\}\left\{e^{\alpha\int\left(\frac{1}{\chi r_y}-\frac{\chi}{r_x}\right)ds}\right\} + c_3 e^{2\rho s} \quad (9) \end{aligned}$$

$$\begin{aligned} a_y &= \left\{c_3\int\left[\left(e^{2\rho s+\alpha\int\left(-\frac{1}{\chi r_y}+\frac{\chi}{r_x}\right)ds}\right)(\alpha+\rho r_x)\right.\right. \\ &\left.\left.\frac{1}{r_x}\right]ds + c_4\right\}\left\{e^{\alpha\int\left(\frac{1}{\chi r_y}-\frac{\chi}{r_x}\right)ds}\right\} \quad (10) \end{aligned}$$

where c_3 and c_4 are arbitrary constants determined by initial values. We solve the integrals in (5) and (6), (9) and (10) to obtain the acceptance evolution without and with correlation, respectively. We launch the beam with $\alpha = 0.00001$ and $\rho = 0.25$ in space-charge dominated regime. $c_1 = 0.22360$ and $c_2 = 0.22360$, $c_3 = 0.44721$ and $c_4 = 0.22360$ are determined by initial values $\tilde{a}_{x0} = \tilde{a}_{y0} = 0.22360$, and $a_{x0} = a_{y0} = 0.22360$. The corresponding evolution of the rms-emittance (1) and the rms-envelope are used. The acceptance evolutions are shown in Fig. 2. $\tilde{a}_x$ and $\tilde{a}_y$, respectively, remain constant with very small oscillations because $\tilde{a}_x$ and $\tilde{a}_y$ are beam area projections derived from Lyapunov functions (2). However, it is observed coupling in a_x and a_y caused for space charge driven core-core resonance together with single-particle resonances [1, 3].This coupling is characterized by the energy exchange between the directions. Energy/acceptance exchange requires resonant coupling. The relaxation of the system is due to a condition of resonance and it may happen that the relaxation stops because there is no resonance anymore. After the energy to have been redistributed among all

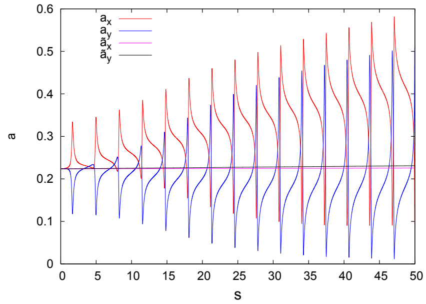

Figure 2: Evolution of the beam acceptance without ($\tilde{a}_x$ and $\tilde{a}_y$), and with (a_x and a_y) correlation between particle position and transverse momentum.

the degrees of freedom, the beam achieve "thermal" equilibration.

To characterize the "thermal" equilibrium we analyze the dynamics of the Lyapunov function (2), and of the temperature and entropy of the beam. The "temperature" T_j of the *j-th* degree of freedom can be expressed as $T_j = \frac{m a_j^2}{K_b r_j^2}$, where m is the mass ions beam and K_b is the Boltzmann's constant [3]. This formula shows that envelope r_j and acceptance a_j variations cause temperature variations. For a system with "non-equilibrium temperatures " that is oscillating anisotropicly around T_{eq}, the equilibrium temperature can be aproximated by the arithmetic average of the T_j :

$$T_{eq} = \frac{m}{2K_b}\left(\frac{a_x^2}{r_x^2} + \frac{a_y^2}{r_y^2}\right) \quad (11)$$

With the equilibrium temperature T_{eq} for the 2-D beam model, the entropy change near thermodynamic equilibrium due to a temperature balancing process may then be written as

$$\frac{dS}{ds} = \frac{K_b}{2}\left[\frac{(T_x - T_y)^2}{T_x T_y}\right] \quad (12)$$

Obviously, the entropy $S(s)$ remains unchanged in the case of temperature equilibrium while increasing during temperature balancing. The basis for the dynamics behaviour of the entropy is the relation between the acceptance, the beam temperature and the envelope. Beam transport without an increase of entropy are thus possible if either the beam stays round throughout its propagation. We launch the beam with $m = 1$, $r_{j0} = 0.43616$ and $a_{j0} = 0,22360$, consequently $T_{j0} = 0.26282$ initially, and integrate the entropy equation (12) up to $s = 200.0$. The dynamics S (bottom), T_{eq} (top) and L (top) are shown in Fig. 3. L is a monotonically decreasing function with respect to s for solutions $\tilde{a}_j$. L becomes stationary in the limit $s \to \infty$. T_{eq} oscillates after steady with small fluctuations and S grows by cascades. The concept of entropy cascade is the key agent in the heating and relaxation of the beam [3]. We observe to have different regimes. There is first a phase of violent relaxation on a time scale $s = 50$ leading to a quasi-stationary state. This phase is followed by a thermalization

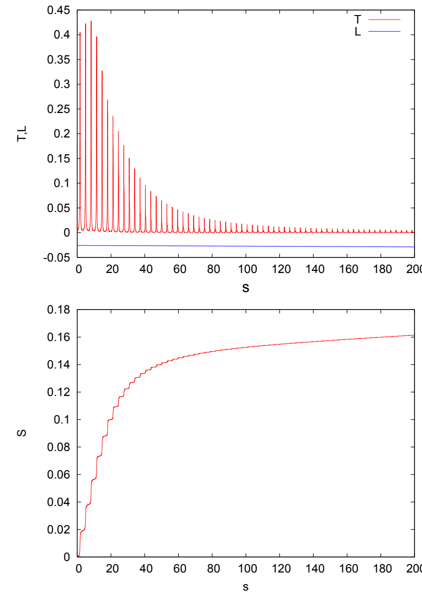

Figure 3: Evolutions of the temperature T_{eq} (top), Lyapunov function L (top) and entropy S (bottom) obtained of the equations (11), (2) and (12), respectively.

leading to the T_{eq} stationary on a time scale $s = 120$ due to the wave-particle interaction [3]. We consider for intermediate times $s = 50$ to $s = 120$ that the distribution function is a quasistationary solution of the Vlasov equation that slowly thermalizies via space-charge. The study this dynamic will be considered in a future work.

CONCLUSIONS

Based on turbulent equipartiton [2] we propose the anisotropic equipartition. In future it is necessary to find experimental evidence of the anisotropic equipartition.

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