ANALYSIS OF ORBITS IN COMBINED FUNCTION MAGNETS

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Abstract

Fixed-Field Alternating-Gradient (FFAG) accelerators span a large range of momenta and have markedly different reference orbits for each momemtum. In the non-scaling (NS) versions proposed for rapid acceleration, the orbits are geometrically dissimilar. In particular, none of the orbits within bending magnets are arcs of circles and this complicates tune calculation. One approach to NS-FFAG design is to employ alternating combined-function magnets[1]. Second generation NS-FFAGs designs attempt to mitigate the variation of betatron tunes; and careful calculation of orbits and tunes is essential. Starting from an analytic magnetic potential for the combined-function magnet, we elucidate expressions for orbit calculation which are second order in the cyclotron motion and arbitrary order in the momentum (no expansion is used).

MAGNET FIELD

We adopt a cylindrical polar coordinate system with origin at the magnet centre of curvature, radius r, azimuthal angle ϕ, and vertical displacement z from the mid plane. Let the curvature at the reference momentum p_c be r_c and the field strength and gradient be B_0 and B_1. Naively, it might be thought that the field components are

$$\mathbf{B} = \left(B_r, B_\phi, B_z\right) = \left(B_1 z, 0, B_0 + B_1\left(r - r_c\right)\right).$$

The curl is zero, but there is a first order divergence: $\nabla\cdot\mathbf{B} = B_1 z/r_c$. A better approximation is needed:

$$\mathbf{B} = \left(\frac{4B_1 r_c^2 z}{\left(r + r_c\right)^2}, 0, B_0 + \frac{2B_1 r_c\left(r - r_c\right)}{\left(r + r_c\right)}\right). \qquad (1)$$

The curl is zero, and there is a 2nd order divergence $\nabla\cdot\mathbf{B} = B_1 z\left(r_c - r\right)/\left(2r_c^2\right)$ which is very small. This field is obtained from a potential function, and hence so may be obtained the pole shape as a function of radius.

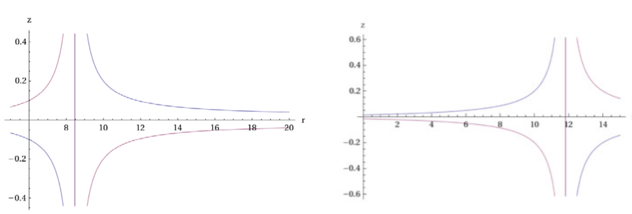

Figure 1: Pole shapes: focusing (LHS), defocusing (RHS). The asymptote denotes the lower and upper range, respectively, for which stable orbits may be found. Blue and red lines denote north and south poles, respectively. B_0= -5 Tesla, B_1=±5 T/m and r_c=10 metre.

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REFERENCE RADII

Although they are not necessarily equilibrium orbits, there are (circular) arc-shaped trajectories of radius r_u at other momenta $p_u = p_c + \Delta p$. Let $\chi = \left(B_0 + B_1 r_c\right)$. The radius is given by the two solutions of a quadratic equation:

$$r_u \simeq r_c + \frac{\Delta p/c}{\chi} - \frac{B_1\left(\Delta p/c\right)^2}{2\chi^3} + \ldots \qquad (2)$$

$$r_u \simeq \frac{-r_c}{\left(1 + 2r_c B_1/B_0\right)}\left[1 + \frac{B_1}{B_0}\frac{\Delta p/c}{\chi}\right] + \frac{B_1\left(\Delta p/c\right)^2}{2\chi^3} + \ldots$$

When χ=0 and Δp=0, the solutions are equal. Hence you cannot take it for granted that $-B_0/B_1$ is distant from r_c. Hence, use Eq. (2) for χ>0 and the 2nd equation for χ<0.

EQUATION OF MOTION

The FFAG is composed of a sequence of combined function magnets (CFMs) with forward and reverse bending and alternating gradients, and also drift spaces. In general, the equilibrium orbit at any momentum does not lie on an arc through the CFM. Instead, the particle moves with a displacement and divergence relative to the reference arc for each momentum. The reference arc is swept out at angular velocity ω_u. To find the equilibrium orbit for given input conditions to the CFM, we take a Cartesian coordinate system $\left(x, s, z\right)$ and embed it in the rotating reference frame. x, s, z are aligned with r, φ, z respectively of the cylindrical system, with the origin $x=0$ at $r=r_u$. When the equations of motion (for perturbations with respect to the rotating cylindrical system) are written in the Cartesian system, the usual Coriollis forces will appear. In addition, one must write the magnetic field components in terms of the new coordinates. The field on the reference arc is:

$$\mathbf{B} = \left(0, 0, B_0 + 2B_1 r_c\left(r_u - r_c\right)/\left(r_u + r_c\right)\right). \qquad (3)$$

The incremental field at a displaced orbit $\left(x, s, z\right)$ is

$$\delta B_x = 2B_1 z - B_1\left(r_u + x\right)z/r_c$$

$$\delta B_s = B_1 s z\left[\frac{2}{r_u} - \frac{1}{r_c}\right] \qquad (4)$$

$$\delta B_z = B_1 s^2\left[\frac{1}{r_u} - \frac{1}{2r_c}\right] + 2B_1 x - \frac{B_1 x}{r_c}\left(r_u - x/2\right)$$

Let $k = \pm B_1\left(2r_c - r_u\right)/\left(B_z r_c\right)$ and $\omega_u = -qB_z/\left(m_0 c\beta\gamma\right)$ where B_z is evaluated at r_u in the mid plane. We introduce the time derivative $\partial x/\partial t \equiv \dot{x} \equiv \beta c x'$ where βc is the particle speed.

Linearized Equations

To first order, the equations of motion are:

$$
\begin{aligned}
&x'' \pm k\omega x - \omega s' = 0 \\
&s'' + \omega x' = 0 \\
&z'' \mp k\omega z = 0
\end{aligned}
\tag{5}
$$

Take positive/negative sign for focusing/defocusing and likewise for k. These coupled equations are exactly soluble, and their solutions form the basis for higher-order expansions. E.g. in a CFM that is horizontally focusing:

$$(x, x', s, s')_t = \mathbf{M}(x, x', s, s')_0$$

where the matrix $\mathbf{M} \times (k+\omega) =$

$$
\begin{bmatrix}
\omega + k\cos ut & (u/\omega)\sin ut & 0 & 1-\cos ut \\
-ku\sin ut & (k+\omega)\cos ut & 0 & u\sin ut \\
\dfrac{k\left[\omega t(k+\omega) - u\sin ut\right]}{k+\omega} & -1+\cos ut & 1 & \dfrac{kt(k+\omega)+u\sin ut}{k+\omega} \\
-k\omega(-1+\cos ut) & -u\sin ut & 0 & k+\omega\cos ut
\end{bmatrix}
$$

and $u = \sqrt{\omega(k+\omega)}$.

Higher Order

To second order, the equations of motion are:

$$
\begin{aligned}
&x'' \pm k\omega x(1+s') - \omega s' \pm \left(k\omega^2/2\right)\left(x^2+s^2\right) = 0 \\
&s'' + \omega x' \pm k\omega(zz' - xx') \pm k\omega^2 sx = 0 \\
&z'' \mp k\omega z(1+s') = 0
\end{aligned}
\tag{6}
$$

Take the plus/minus sign for horizontally focusing/defocusing field gradient.

Green's Function Solution

The procedure for solution of these equations is to move the 2nd order terms to the right of the equality, to substitute the known solutions to the linear equations, and then to treat the system as if there were known driving terms on the right, and then to solve for (x, x', s, s') using the Green's function for the linear system. Such a solution is not self-consistent. However, the errors are of 3rd order and will be small provided that (x, x', s, s') remain small. One advantage of this approach is that the Green's functions can be obtained analytically in closed form in terms of trigonometric and hyperbolic functions. The expressions are too lengthy to record here. However, they have been incorporated into a computer program[2] which evaluates them automatically for arbitrary values of all parameters.

It is worth pointing out that in the case of a pure bending magnet, k=0, that the linear equations are exact and the particle speed is conserved in the matrix solution above. However, when a field gradient is present, the linearized equations, although the matrix has unity determinant, do not conserve the particle velocity. A consideration of the Lorentz forces explains this situation. For the flat field, the forces are $\mathbf{F} = \mathbf{v} \times \mathbf{B}$. When the gradient is added, the force becomes $(\mathbf{v} + \delta\mathbf{v}) \times (\mathbf{B} + \delta\mathbf{B})$, and discarding the 2nd order terms $\delta\mathbf{v} \times \delta\mathbf{B}$ amounts to destroying the energy conserving property of motion in a magnetic field. This notion is explored in the following examples which compare the results of particle tracking through CFMs with linear and 2nd-order transformations of the input coordinates.

EXAMPLES

When graphed, we can compare second order results with those obtained using linearized equations of motion. For definiteness, we take the case of a focusing CFM with B_0= 5 Tesla and B_1= 5 T/m. The reference momentum is p_c= 15 GeV/c and the radius of curvature r_c= 10.00692 metres. A 16 GeV/c proton is injected with an orbit offset x= 1cm compared with the arc radius r_u= 10.06737. Somewhat artificial, but a good test, we consider a CFM with 360^o bending angle and three orbits (i.e. a total bend of 10 radians.) After only three orbits around the CFM, there is apparent a discrepancy between the two results. We see clearly (Figs. 2 and 3) that the second order equations are able to account for the longer trajectory of a "wiggling" orbit which causes an off-arc particle to slip behind the reference particle. The linearized motion shows no sign of this occurring and also shows a greater variation in the total speed of the particle, see Fig.4.

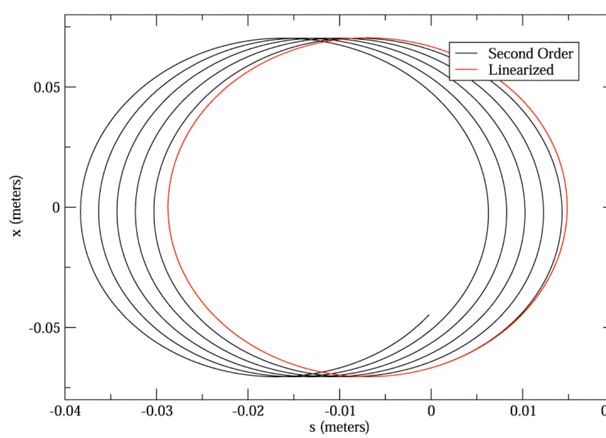

Figure 2: Parametric plot of the position coordinates of with respect to the reference particle.

It may be wondered why the particle slips in azimuth s, but not radius x. In the case of a pure bending field, the equations of motion for x and s are symmetric, $x'' - \omega s' = 0$ and $s'' + \omega x' = 0$. However, for the case of a gradient field, the x motion is supplemented by a restoring force [see the linearized equations (5) above] that tends to confine the motion and makes it resistant to perturbation by the 2nd order terms. No such restoring force appears in the s motion, and so it is more easily perturbed. Hence when the second order term $\delta\mathbf{v} \times \delta\mathbf{B}$ is added, it will influence more strongly the motion in s than that in x. As the gradient is reduced in magnitude, so the influence on s/x will become less/more pronounced, respectively. In the limit $k \to 0$, there is no restoring force; but there is also no perturbation because $\delta\mathbf{B} = 0$.

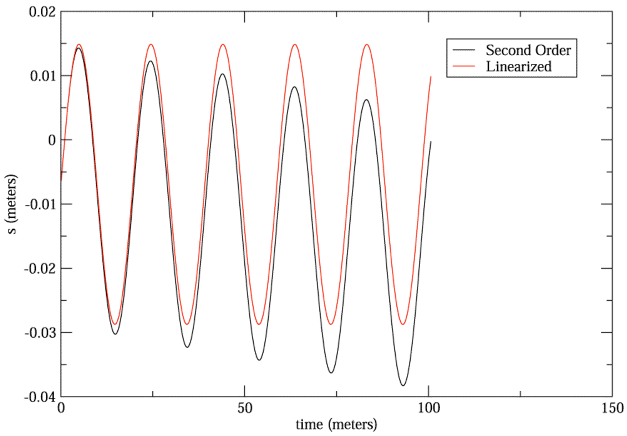

Figure 3: Longitudinal displacement s versus time.

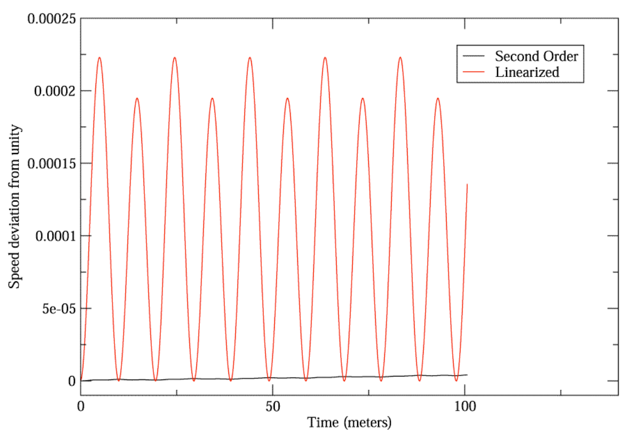

Figure 4: Deviation of speed from unity versus time.

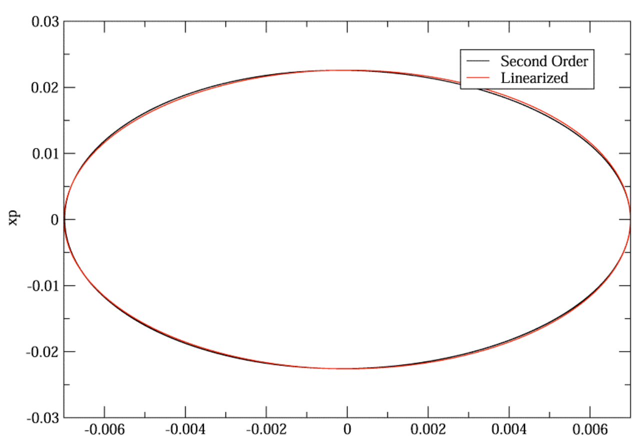

Figure 5: Parametric plot of the velocity coordinates x' versus s' with respect to the reference particle.

This line of reasoning implies that for a horizontally defocusing CFM, it is the x motion which will be more strongly affected by the perturbing forces, because there is an anti-restoring force, $-k\omega x$. Repeating the particle tracking exercise for a defocusing CFM, when comparing second and first order formulations, one expects to see the larger discrepancy to appear in *x(t)*, rather than *s(t)*, because the negative restoring force in the radial equation of motion makes it more susceptible to perturbation by the second order terms $\delta\mathbf{v}\times\delta\mathbf{B}$. This expectation is confirmed in figures 6 and 7: position versus time graphs of motion in a defocusing CFM with magnetic field gradient B_1 of 1 Tesla per meter and azimuthal angular extent of 1.0 radian.

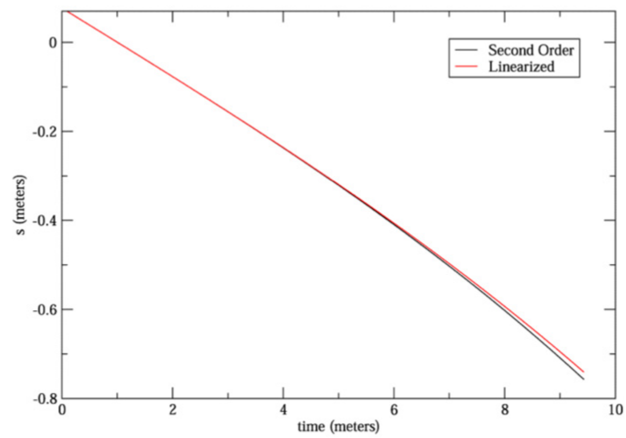

Figure 6: Longitudinal displacement s versus time in a defocusing CFM.

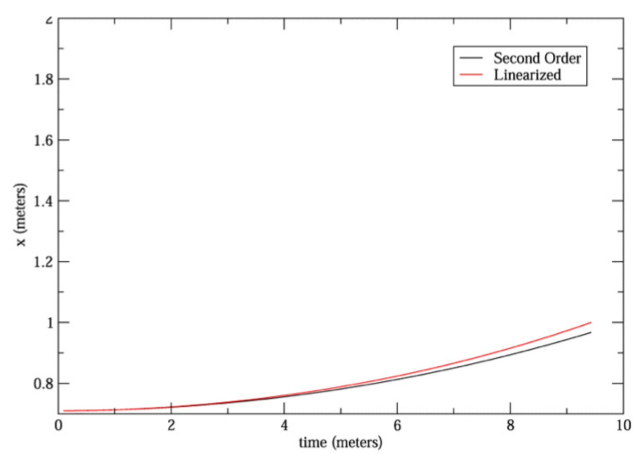

Figure 7: Radial displacement x in a defocusing CFM.

REFERENCES

[1] S. Koscielniak: Formulae for off-momentum closed orbits … in lattice cells containing sectoral combined function magnets, TRI-DN-06-04, June 2006.

[2] T. Mackenzie: Analysis of motion in CFM, U. Prince Edward Island Co-op Student program, Dec. 2008.