AG FOCUSING IN THE THOMAS CYCLOTRON OF 1938

M.K. Craddock*, University of British Columbia and TRIUMF

Abstract

It is sometimes asserted that Thomas's proposal to provide additional axial focusing in cyclotrons by introducing an azimuthal variation in the magnetic field (to enable them to operate isochronously at relativistic energies) was an early example of alternating-gradient focusing. While Thomas cyclotrons certainly exhibit alternating field gradients, it is shown that the alternating focusing produced is very much weaker than the edge focusing (everywhere positive) arising from orbit scalloping.

INTRODUCTION

In attempting to explain the focusing mechanism of Thomas and other radial-sector cyclotrons it is occasionally claimed (even by very respectable authors) that it is a form of alternating-gradient (AG) focusing. Thus Livingston and Blewett, in their classic text, *Particle Accelerators* [1], say (p. 640):

> "Thomas's proposal of sector focusing was a special case of the general theory of AG focusing." …and make more specific comments, quoted below.

In a more recent textbook, Bryant and Johnsen [2] write:

> "[Thomas's cyclotron] was a forerunner of alternating gradient focusing……".

Even Courant [3] supports this view:

> "Actually, strong focusing had also been anticipated by L.H. Thomas in 1938. He had devised a modification of the cyclotron that would have strictly constant orbit frequency and would achieve the necessary orbit stability by means of azimuthal field variation - indeed, a (weak) version of alternating-gradient focusing."

As a final example I quote from a 2008 poster for *Reviews of Accelerator Science and Technology*:

> **"FFAG:** 1956: The first Fixed-Field Alternating-Gradient accelerator is commissioned…..... An earlier variation is conceived by Llewellyn Thomas in 1938."

While it's true that Thomas did introduce alternating gradients into accelerators, they were incidental and produced negligible focusing. The source of Thomas focusing is the azimuthal alternation of field *strength*, which distorts the orbits. When a myth continues to circulate, it seems worth trying to lay it to rest – and this paper will attempt to do that.

"STRONG" FOCUSING FROM ALTERNATING LENSES

The basic feature needed for so-called "strong" focusing is a string of lenses of equal and opposite polarities:

… F - D - F - D - F - D - F - D - F - D …

whose overall effect is focusing because the deflexions in thin lenses are proportional to displacement and therefore on average greater at Fs than at Ds. Note that it's a differential or 2nd-order effect – the focusing is much weaker than achievable with FFFFFF lenses of the same strength. Calling it "strong" focusing is in fact somewhat misleading, as "strong" better describes the lenses than the overall effect. "AG focusing" can also mislead, as the same effect can be produced by other entities than gradients – in particular by magnet edges, as we shall see. I shall use the term "alternating focusing".

Asymmetric Alternating Focusing

To deal with focusing in cyclotrons it will be useful to consider a F0G0 cell, where drifts are of equal length *a*, but the lenses have unequal focal powers *F* and *G*. Then the transfer matrix:

$$M = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -G & 1 \end{pmatrix} \quad (1)$$

and from its trace we find:

$$4\cos^2\left(\mu/2\right) = (2 - aF)(2 - aG) \quad (2)$$

- so, in *F*-*G* space, curves of phase advance μ = *constant* are hyperbolae (Figure 1).

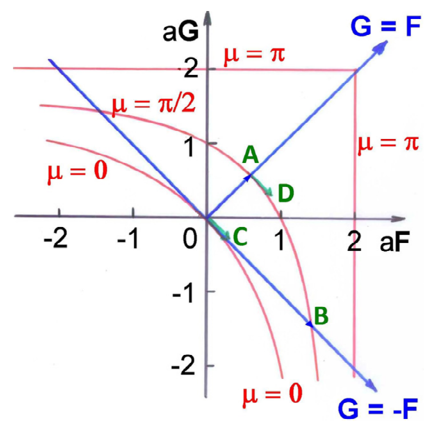

Figure 1: Curves of constant phase advance μ in *F*-*G* space.

The figure shows, as expected, that much stronger lenses are needed to reach $\mu = \pi/2$ at B (alternating lenses, *G* = -*F*) than at A (equal lenses, *G* = *F*). But we also see that if $G \neq |F|$, weak alternating lenses produce little extra focusing, either alone (C) or as modulation on strong positive focusing (D)

WHAT IS THOMAS FOCUSING?

Thomas is best known for Thomas spin precession and the Thomas-Fermi statistical model of the atom, but in 1938 he responded to a claim by Bethe and Rose [4] that relativity would prevent cyclotrons reaching energies >20 MeV with a proposal [5] that has left a lasting legacy – the many small cyclotrons used for isotope production.

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The problem was that the basic cyclotron equation for the angular frequency of an ion of mass m and charge q in uniform magnetic field B:

$$\omega = \frac{qB}{m} = const. \tag{3}$$

would be violated as the ion's energy, and therefore its effective mass, rose. Increasing B with radius r would recover constant ω, but then axial stability would be lost, since:

$$\nu_z^2 = \frac{r}{B_{av}} \frac{dB_{av}}{dr} = -\beta^2\gamma^2 \tag{4}$$

Thomas's idea was to use an azimuthally varying field:

$$B(\theta) = B_{av}(1 + f\cos N\theta) \tag{5}$$

to produce a non-circular "scalloped" orbit. The radial v_r and azimuthal B_θ components then yield an axial force component $F_z(\theta)$ that acts in a focusing sense at all azimuths, with the result:

$$\nu_z^2 = -\beta^2\gamma^2 + \tfrac{1}{2} f^2. \tag{6}$$

Thus axial focusing can be recovered for relativistic ions, provided f is large enough. (He also pointed out that the motion would be unstable for $N < 3$). Note that Thomas focusing differs from AG focusing not only in being uniform in sign, but also in stemming from $v_r B_\theta$, not $v_\theta B_r$.

Thomas's idea was first realized by Richardson *et al.* [6], who achieved $\beta = 0.5$ with an electron model at Berkeley in 1950 (Figure 2). Note the harmonically shaped poles!

Figure 2: Electron model Thomas cyclotron.

Radial-Sector Cyclotrons

For modern radial-sector isochronous cyclotrons with flat "hill" pole pieces, Thomas focusing becomes an edge-focusing effect $\propto \tan\kappa$, where κ is the edge-crossing or "Thomas angle" (Figure 3) - itself proportional to the mean square deviation in $B(\theta)$- the "flutter":

$$F^2 \equiv \left\langle \left(\frac{B(\theta) - B_{av}}{B_{av}} \right)^2 \right\rangle \tag{6}$$

The axial tune is given by:

$$\nu_z^2 = -\beta^2\gamma^2 + F^2. \tag{7}$$

The lower part of Figure 3 clearly shows that every edge acts as an F lens.

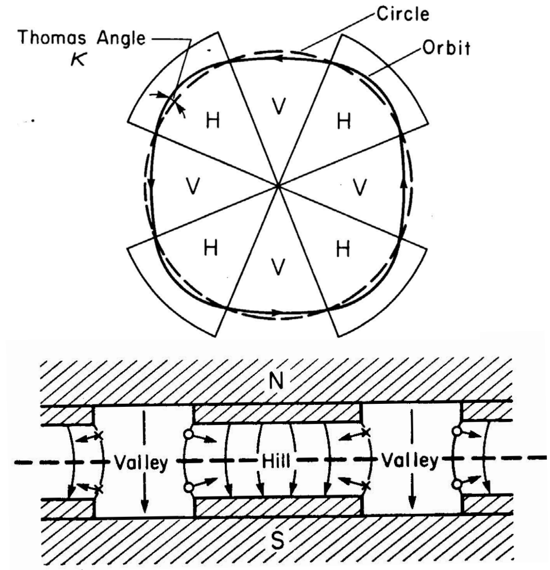

Figure 3: A radial-sector cyclotron, showing orbit scalloping and the Thomas focusing at the sector edges.

Misconceptions

Livingston & Blewett's statement quoted earlier:

> "Thomas's proposal of sector focusing was a special case of the general theory of AG focusing"

seems to have arisen from a misunderstanding of this last point, as they make explicit on page 642:

> "The "Thomas" force F_z, which provides vertical stability in this field, is due to the interaction of the radial component of momentum mv_r with the azimuthal component of field B_θ. The direction of this force alternates as the particle moves around the orbit, from convergence to divergence about the median plane. This rapid alternation of focusing and defocusing forces provides the same type of stability which is characteristic of the forces in AG focusing, resulting in net convergence."

The first sentence is correct, but not the remarks about alternation. Their confusion perhaps arose because alternating focusing *is* employed in isochronous cyclotrons with spiral sectors - but *not* in Thomas or radial-sector cyclotrons.

ALTERNATING FOCUSING IN FFAGS AND CYCLOTRONS – SPIRAL SECTORS

Kerst [7] first suggested using spiral sectors to increase the axial focusing in FFAGs and cyclotrons. A spiral angle ε ($> \kappa$) leads to crossing angles of $\kappa + \varepsilon$ at one edge of each sector (i.e. a strong F lens) and $\kappa - \varepsilon$ at the other (*i.e.* a less strong D lens) – see Figure 4. The focal powers become:

$$\frac{1}{f_z} = \pm \frac{e\left(B_h - B_v\right)}{mv} \tan(\varepsilon \pm \kappa) \tag{8}$$

and overall:

$$\nu_z^2 = -\beta^2\gamma^2 + F^2\left(1 + 2\tan^2\varepsilon\right) \tag{9}$$

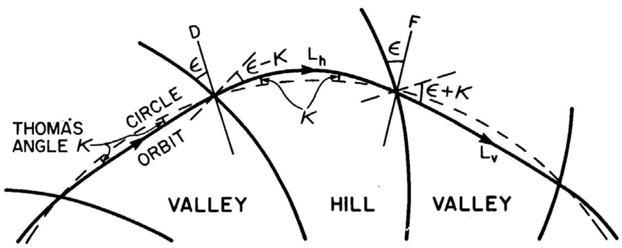

Figure 4: Edge crossing angles in a spiral-sector cyclotron.

Spiralling is now used for most isochronous cyclotrons over 40 MeV. The powerful $2\tan^2\varepsilon$ term has enormously increased the energies attainable. Thus the TRIUMF H$^-$ cyclotron ($\varepsilon \le 70°$) can deliver 250 μA beams at 70 – 520 MeV, and the PSI proton cyclotron ($\varepsilon \le 35°$) 2 mA beams at 590 MeV – up to a thousand times more intense than were previously available for π, μ, neutron and radioactive ion production.

ALTERNATING GRADIENTS IN RADIAL-SECTOR CYCLOTRONS

But if $B(\theta)$ oscillates in a radial-sector cyclotron, surely $\partial B/\partial r$ must too! Indeed it does, and produces some alternating focusing – but, as we shall see, its contribution is negligible.

Consider a hard-edge radial-sector cyclotron, with $\omega = m_0 B_c/q$, hills and valleys of equal length, and:

$$B(\theta) = B_\pm = \gamma B_c(1 \pm \beta\gamma) \qquad (10)$$

so that the flutter $F^2 = \beta^2\gamma^2$ is just sufficient to cancel the isochronous defocusing. To simplify the algebra, we consider the axial focusing induced by the radial gradients (focusing in the valleys, defocusing in the hills) separately from that produced by the edges. Then the axial transfer matrix for the cell can be written:

$$M_z = \begin{pmatrix} \cos\phi_- & k_-^{-1}\sin\phi_- \\ -k_-\sin\phi_- & \cos\phi_- \end{pmatrix}\begin{pmatrix} \cosh\phi_+ & k_+^{-1}\sinh\phi_+ \\ k_+\sinh\phi_+ & \cosh\phi_+ \end{pmatrix} \qquad (11)$$

where the phase advances, $\phi_\pm = k_\pm \ell/2$, with:

$$\ell = \frac{2\pi\beta c}{N\omega} \qquad k_\pm \equiv \sqrt{\frac{\partial B_\pm/\partial r}{B\rho}} = k\sqrt{\lambda \pm 1}$$

$$k = \frac{\gamma\omega}{c} \qquad \lambda \equiv \frac{\gamma}{\beta}(2\gamma^2 - 1)$$

Expanding Trace M_z, we find:

$$4\sin^2\left(\frac{\mu_z}{2}\right) = -k^2\ell^2 + \left(\frac{\lambda^2 - 4}{48}\right)k^4\ell^4 \qquad (12)$$

so the gradient contribution to the vertical focusing is:

$$\Delta_{grad}(\nu_z^2) = -\beta^2\gamma^2 + \left(\frac{\pi^2\lambda^2}{12N^2}\right)\beta^4\gamma^4 \qquad (13)$$

where the second term, always positive, describes the AG focusing effect. In the Thomas regime, where $\beta << 1$, $\gamma^2 \rightarrow 1+\beta^2$ and $\lambda^2\beta^4\gamma^4 \rightarrow \beta^2 + 7\beta^4$, so that:

$$\Delta_{grad}(\nu_z^2) = -\beta^2 + \frac{\pi^2}{12N^2}\beta^2 - \left(1 - \frac{7\pi^2}{12N^2}\right)\beta^4 \qquad (13)$$

i.e. the AG contribution is at most $\pi^2/12N^2$ of that provided by Thomas edge focusing to cancel the $-\beta^2$ term. For the worst allowable case ($N = 3$), this amounts to less than 10%. And referring back to Figure 1 it's clear that the net effect on phase advance and tune is smaller still.

CONCLUSIONS

A table comparing the relevant features of a Thomas cyclotron with those a radial-sector scaling FFAG will form a useful summary of the points that have been made:

Table 1: AG focusing in Thomas cyclotrons and FFAGs.

	Thomas cyclotron	FFAG / AGS
Periodic $B(\theta)$	Yes	Yes / No
Alternating $\partial B/\partial r$	Yes	Yes
Axial force	$q v_r B_\theta$	$q v_\theta B_r$
Lens pattern	FFFFFFFF	FDFDFDFD
Edge focusing	Dominant	Negligible
AG focusing	Negligible	Dominant

Alternating field gradients occur in both accelerator types, but that similarity is a superficial one: the mechanism of Thomas focusing is quite different from that of AG focusing. I conclude that:

- Thomas's proposal was *not* "a special case of the general theory of AG focusing"
- strong focusing had *not* "been anticipated by L.H. Thomas"
- Thomas focusing was *not* "a (weak) version of alternating-gradient focusing."
- Thomas's cyclotron did *not* represent "An earlier variation" of the FFAG."

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