TRANSVERSAL THRESHOLD FOR MODULATIONAL INSTABILITY IN LASER-PLASMA SYSTEMS*

A. Bonatto†, R. Pakter, F. B. Rizzato
Instituto de Física, Universidade Federal do Rio Grande do Sul, Caixa Postal 15051,
91501-970 Porto Alegre, Rio Grande do Sul, Brasil.

Abstract

In the present analysis we study the self consistent propagation of intense laser pulses in a cold relativistic ideal-fluid underdense plasma, with particular interest in how the envelope dynamics is affected by the plasma frequency. Analysis of the linear system associated with the chosen model shows the existence of thresholds that can lead propagating pulses to distinct modulational instabilities, according to the relation between its transversal wave vector and the plasma frequency.

INTRODUCTION

Intense laser pulses propagating in a plasma can create (through action of the nonlinear ponderomotive force) wakefields that, under appropriate conditions, will trap electrons and provide them a high gradient of acceleration. In last years much progress has been done in this area, from Tajima & Dawson [1] computational simulations to recent Leemans et al. [2] experiments. Stability analysis of laser-plasma systems plays an important role to have a deeper understanding about the processes involved with the creation and destruction of these wakefields.

One-dimensional models used previously [3, 4, 5] for analytical and computational works were studied from the perspective of nonlinear dynamics [6, 7], and our purpose is to extend this perspective to models with transversal and longitudinal dynamics. We start from a model used by Mora & Antonsen [9], using it to obtain a pair of coupled equations for two perturbed quantities: the plasma density $\bar{n}$ and the transversal vector potential amplitude $A_\perp$. From these equations it is possible to deduce a dispersion relation for the propagation of waves in the plasma. Solutions of dispersion relation show the existence of thresholds for modulational instabilities, which depends on the relation between $K_\perp^2$ and ω_p^2.

MODEL

Considering the self consistent propagation of laser pulses in a cold relativistic plasma, we assume that the electrons interact with the electric field in two separated ways: first, jittering due to the high frequency laser field and, second, creating a nonlinear wake following the laser pulse as a response to its low frequency ponderomotive potential. This assumption is reasonable since the plasma is underdense; the laser frequency is much greater than all the other characteristic times in the system. That is, $\bar{\omega}_p \ll \omega_0$ and $r_L \gg 1/k_0$, where $\bar{\omega}_p \equiv (4\pi q^2 n_0/m)^{1/2}$ is the plasma frequency based in the background density n_0, the charge q and the mass m of the electrons.

Following the model proposed by Mora & Antonsen [9] we start from a first-order linearized equation of motion, which considers the combined electrostatic and ponderomotive potentials (associated to the electric wakefield and to the jitter of electrons in the laser field respectively),

$$\frac{\partial}{\partial t}(\bar{\gamma} m \bar{\mathbf{v}}) = -q\nabla\bar{\phi} - \frac{q^2}{2\bar{\gamma} m c^2}\nabla\overline{|\tilde{\mathbf{A}}_\perp|^2}\,, \qquad (1)$$

where tilde and overbar are used to identify rapidly and slowly varying components respectively; $\bar{B}$ field was neglected as it has been shown [8] that it is of higher order in the small parameter $\bar{\omega}_p/ck_0$, to be used in the subsequent analyses. Radiation is written in terms of the circularly polarized high frequency vector potential $\hat{\mathbf{A}}_\perp$ as an envelope modulating a plane wave travelling at the speed of light,

$$\tilde{\mathbf{A}}_\perp = \hat{\mathbf{A}}_\perp(z, \mathbf{x}_\perp, t)exp[ik_0\zeta] + c.c\,, \qquad (2)$$

where $\zeta = z - ct$ and k_0 is the wave number of the plan wave. Expanding the envelope complex amplitude as $\hat{A}_\perp = A_0 + A_\perp$ (with $A_\perp \ll A_0 =$ constant) we can calculate $\bar{\gamma}$ with the lowest order of this expansion,

$$\bar{\gamma} = \left[1 + \frac{q^2 A_0^2}{m^2 c^4}\right]^{1/2}, \qquad (3)$$

and rewrite the term involving the vector potential in Eq. (1) as

$$\overline{|\tilde{\mathbf{A}}_\perp|^2} = 2|\hat{A}_\perp|^2 = 2A_0(A_\perp + A_\perp^*)\,. \qquad (4)$$

Finally, expanding $n = n_0 + \bar{n}$ in the continuity equation (with $\bar{n} \ll n_0 =$ constant) and using Poisson equation $\nabla^2\bar{\phi} = -4\pi q\bar{n}$ we obtain the following equation for the density:

$$\left(1 + \frac{\bar{\gamma}}{\bar{\omega}_p^2}\frac{\partial^2}{\partial t^2}\right)\bar{n} = \frac{A_0}{4\pi m\bar{\gamma} c^2}\nabla^2(A_\perp + A_\perp^*)\,. \qquad (5)$$

For the envelope, expanding $A_\perp$ n and neglecting higher order terms we have

$$\left(\frac{2ik_0}{c}\frac{\partial}{\partial t} + \nabla_\perp^2\right)A_\perp = \frac{\bar{\omega}_p^2}{n_0\bar{\gamma} c^2}(n_0 A_\perp + \bar{n} A_0)\,. \qquad (6)$$

* Work supported by CNPq, Brazil, and AFOSR (FA9550-06-1-0345).
† abonatto@if.ufrgs.br

Writing the relevant complex quantities as $A_\perp = |A_\perp| exp[\bar{\mathbf{K}} \cdot \mathbf{r} - \bar{\Omega} t]$ and $\bar{n} = |\bar{n}| exp[\bar{\mathbf{K}} \cdot \mathbf{r} - \bar{\Omega} t]$, the system of Eqs. (5) and (6) can be used to obtain the dispersion relation for waves (laser pulses, wakefields) propagating in the plasma:

$$\Omega^2 = \omega_p^2 + 4\omega_p^2 \alpha^2 K^2 \frac{(K_\perp^2 - \omega_p^2)}{\left[4\Omega^2 - (K_\perp^2 + \omega_p^2)^2\right]}\,, \qquad (7)$$

where

$$\Omega^2 \equiv \frac{\bar{\Omega}^2}{c^2 k_0^2} \quad , \quad \omega_p^2 \equiv \frac{\bar{\omega}_p^2}{\bar{\gamma} c^2 k_0^2}$$
$$\mathbf{K} \equiv \frac{\bar{\mathbf{K}}}{k_0} \quad , \quad \alpha^2 \equiv \frac{q^2 A_0^2}{\bar{\gamma}^2 m^2 c^4}\,.$$

From Eq. (7),

$$\Omega = \pm \frac{1}{2\sqrt{2}} \sqrt{\delta_r \pm \sqrt{\delta_r^2 - \delta_i}}\,, \qquad (8)$$
$$\delta_r \equiv \left[4\omega_p^2 + (K_\perp^2 + \omega_p^2)^2\right]\,, \qquad (9)$$
$$\delta_i \equiv 16\omega_p^2 \left[(K_\perp^2 + \omega_p^2)^2 - 4\alpha^2 K^2\right]\,. \qquad (10)$$

STABILITY ANALYSIS

As shown in Eq. (8), the solution to the obtained dispersion relation, pulses will propagate in the plasma with a frequency Ω, which under determined circumstances can be purely real (and thus there is propagation without instabilities), have real and imaginary components (there is propagation together with instabilities) or be purely imaginary (there are instabilities and propagation is not possible). In order to study such instabilities, it is necessary to analyze the regimes where the frequency is a complex quantity. Since that $\delta_r, \delta_i \in \mathbb{R}$, this can be done determining when it is possible to have a negative quantity in any of the square roots in the Ω solutions.

Particularly for the solution of Eq. (8) with the minus signal inside the square root, the following condition determines if Ω is a real or a complex quantity:

$$\Omega_- \sim \sqrt{\delta_r - \sqrt{\delta_r^2 - \delta_i}} \in \mathbb{C} \;\Rightarrow\; \delta_i < 0\,. \qquad (11)$$

Another condition, valid for all solutions of Eq. (8), can lead Ω to be imaginary:

$$\Omega \sim \sqrt{\delta_r \pm \sqrt{\delta_r^2 - \delta_i}} \in \mathbb{C} \;\Rightarrow\; \delta_r^2 < \delta_i\,. \qquad (12)$$

Solving Eqs. (11) and (12) as functions of α^2 we can establish thresholds (critical values of this parameter) which separate regular from unstable dynamics:

$$\alpha_{c1}^2 \equiv \frac{\left(K_\perp^2 + \omega_p^2\right)^2}{4\left(K_\parallel^2 + K_\perp^2\right)\left(K_\perp^2 - \omega_p^2\right)}\,, \qquad (13)$$

$$\alpha_{c2}^2 \equiv -\frac{\left[\left(K_\perp^2 + \omega_p^2\right)^2 - 4\omega_p^2\right]^2}{64\omega_p^2\left(K_\parallel^2 + K_\perp^2\right)\left(K_\perp^2 - \omega_p^2\right)}\,. \qquad (14)$$

It is worth to observe that the signals of Eqs. (13) and (14), which are always opposites, are determined by the difference between the squared transversal wave number and the squared plasma frequency:

$$K_\perp^2 > \omega_p^2 \quad \Rightarrow \quad \alpha_{c1}^2 > 0\,,\; \alpha_{c2}^2 < 0\,,$$
$$K_\perp^2 < \omega_p^2 \quad \Rightarrow \quad \alpha_{c1}^2 < 0\,,\; \alpha_{c2}^2 > 0\,.$$

Figure (1) shows α_{c1}^2 and α_{c2}^2 behaviors as functions of $K_\perp^2/\omega_p^2$ for a fixed value of ω_p^2.

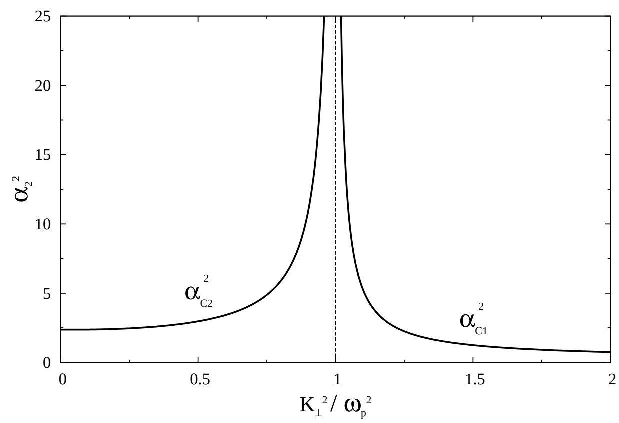

Figure 1: thresholds α_{c1}^2 and α_{c2}^2, plotted as functions of the ratio $K_\perp^2/\omega_p^2$ for $\omega_p^2 = 0.10$; if $K_\perp^2/\omega_p^2 < 1$, α_{c1}^2 is negative and α_{c2}^2 is positive; if $K_\perp^2/\omega_p^2 > 1$, signals are the opposite.

Definition of $\alpha^2 \equiv q^2 A_0^2/\bar{\gamma}^2 m^2 c^4$ shows that this parameter is related with A_0, the term of lowest order of the vector potential amplitude (which is constant and real), and other real squared quantities. For this reason, negative values of α^2 have no physical meaning.

As $\alpha \in \mathbb{R} \Rightarrow \alpha^2 \geq 0$, it is possible to see that, for given values of ω_p^2 and $K_\perp^2$, regimes with $K_\perp^2 > \omega_p^2$ have a threshold like α_{c1}^2, with an absolute instability when it is crossed. Figure (2), plotted for $K_\perp^2 = 0.2$ and $\omega_p^2 = 0.1$, shows the real and imaginary parts of Ω as functions of α^2 ($\alpha_{c1}^2 \approx 0.750$).

Regimes with $K_\perp^2 > \omega_p^2$ are illustrated in Figure (3), plotted for $K_\perp^2 = 0.1$ and $\omega_p^2 = 0.2$ ($\alpha_{c1}^2 \approx 1.969$). Before α_{c2}^2, the frequency Ω is real and there are two stable modulational modes that couple at α_{c2}^2; After the threshold, Ω is complex and the propagation of waves in the plasma becames unstable.

CONCLUSION

In this work we have studied the self consistent propagation of intense laser pulses in a cold relativistic plasma using a model that contemplates longitudinal and transversal dynamics.

Solving a set of two coupled equations we were able to find a dispersion relation and to obtain the possible solutions for the frequency Ω. These solutions allowed us to verify under which conditions $\mathrm{Im}(\Omega) \neq 0$ and to find and express the existing thresholds for modulational instabilities, α_{c1}^2 and α_{c2}^2, as functions of the transversal wave vector of the propagating pulse and the plasma frequency.

We have analyzed the behavior of the thresholds as functions of the ratio $K_\perp^2/\omega_p^2$, which have opposite signals and that only positive values have physical meaning.

For regimes with $K_\perp^2 > \omega_p^2$, the threshold α_{c1}^2 separates regular dynamics from an absolute instability: the real component of Ω vanishes as α^2 goes towards its critical value and, after that, Ω has only imaginary components; at this point, there is no propagation at all.

For regimes with $K_\perp^2 < \omega_p^2$, the instability possibly is a convective one: before the threshold, Ω has two real branches that couple when α_{c2}^2 is reached; after that, Ω has both (real and imaginary) components and instability is present.

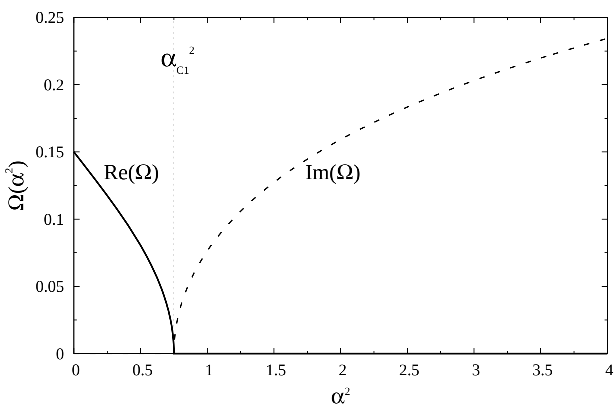

Figure 2: for $K_\perp^2 > \omega_p^2$ the frequency Ω starts as a real quantity that vanishes when $\alpha^2 \to \alpha_{c1}^2$; beyond this threshold waves cannot propagate: there are only unstable modes that keep growing as $\alpha^2 > \alpha_{c1}^2$.

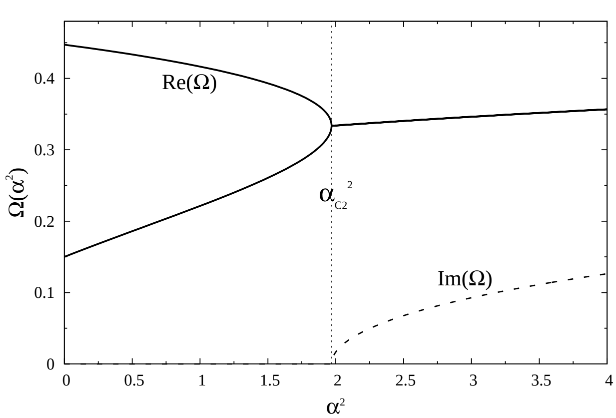

Figure 3: for $K_\perp^2 < \omega_p^2$ the frequency Ω has two stable branches that couple when α_{c2}^2 is reached. Beyond this threshold, which is the onset of unstable mode, Ω is complex and instability growth can be measured by its imaginary component.

REFERENCES

[1] T. Tajima, J. Dawson, Phys. Rev. Lett. 43 (1979) 267.

[2] W.P. Leemans et al., Nature (London) 2 (2006) 696.

[3] V.A. Kozlov et al., JETP (Engl. Transl.) 76 (1979) 148.

[4] U.A. Mofiz , U. de Angelis, J. Plasma Phys. 33 (1985) 107.

[5] D. Farina, S. Bulanov, Phys. Rev. Lett. 86 (2001) 5289.

[6] A. Bonatto et al., J. Plasma Phys. 72 (2005) 179.

[7] A. Bonnato et al., Chaos Sol. & Fractals 32 (2007) 951.

[8] F.B. Rizzato, J. Plasma Phys. 40 (1988) 298.

[9] P. Mora and T.M. Antonsen Jr., Phys. Plasmas 4 (1997) 217.