LANDAU DAMPING WITH HIGH FREQUENCY IMPEDANCE*

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Abstract

Coupled bunch longitudinal stability in the presence of high frequency impedances is considered. A frequency domain technique is developed and compared with simulations. The frequency domain techniqe allows for absolute stability tests and is applied to the problem of longitudinal stability in RHIC with the new 56 MHz RF system.

THEORY

The problem of bunched beam longitudinal stability has been discussed many times [1–12]. In the present treatment it is shown that one can extend dispersion integral techniques to longitudinal modes with complicated internal bunch structure. We assume M identical equally spaced bunches. Let θ denote the azimuth, which increases by 2π each turn, T_0 be the synchronous revolution period and $\omega_0 = 2\pi/T_0$ be the angular revolution frequency. Take $\phi = \theta - \omega_0 t$ as the longitudinal coordinate and consider a driving voltage $\hat{V}\exp(i(k_0M+s)\phi - i\Omega t)$ where s is the coupled bunch mode number and Ω is the fractional drive frequency. In amplitude angle coordinates $\phi = r\sin\psi$ and the Vlasov equation reads

$$-i\Omega F_1 + \omega_s(r)\frac{\partial F_1}{\partial\psi} = \frac{\partial H_1}{\partial\psi}\frac{dF_0}{dr} \tag{1}$$

where $F_0(r) + F_1(r,\psi)\exp(-i\Omega t)$ is the normalized distribution function for the first bunch, $\int F_0(r)2\pi r dr = 1$. The perturbation hamiltonian is due to the applied voltage and the beam induced voltage

$$H_1 = \frac{\bar{\eta}q\omega_0}{2\pi r\omega_{s0}}\left\{\frac{\hat{V}e^{i(k_0M+s)r\sin\psi}}{i(k_0M+s)} - \sum_{k\neq 0}\frac{\rho_k Z_k}{i(kM+s)}e^{i(kM+s)r\sin\psi}\right\} \tag{2}$$

where $\bar{\eta} = \omega_0\eta/(\beta^2E_0)$, η is the frequency slip factor, $\beta = v/c$, E_0 is the average particle energy, q is the charge per particle, ω_{s0} is the small amplitude synchrotron frequency, $Z_k = Z[\omega_0(kM+s)+\Omega]$, and

$$\rho_k = \frac{q\omega_0 NM}{2\pi}\int r dr d\psi F_1(r,\psi)e^{-i(kM+s)r\sin\psi}, \tag{3}$$

where there are N particles per bunch.

To solve the system take

$$F_1(r,\psi) = \sum_{\ell\neq 0} R_\ell(r)e^{i\ell\psi} \tag{4}$$

where ℓ is the synchrotron mode number. Insert eq (4) into eq (1), multiply by $\exp(-im\psi)d\psi/2\pi$ and integrate over ψ,

$$\begin{aligned} i\left[m\omega_s(r) - \Omega\right]R_m(r) &= \oint \frac{d\psi}{2\pi}e^{-im\psi}\frac{\partial H_1}{\partial\psi}\frac{dF_0}{dr} \\ &= imF_0'\oint\frac{d\psi}{2\pi}e^{-im\psi}H_1(r,\psi) \\ &= F_0'\frac{\bar{\eta}qm\omega_0}{2\pi r\omega_{s0}}\left\{\frac{\hat{V}J_m[(k_0M+s)r]}{k_0M+s}\right. \\ &\left. -\sum_{k\neq 0}\frac{\rho_k Z_k}{kM+s}J_m[(kM+s)r]\right\} \end{aligned} \tag{5}$$

Now define

$$C_{m,p} = \int_0^{\hat{\phi}} r dr R_m(r)J_m[(pM+s)r],$$

so that

$$\rho_k = q\omega_0 NM\sum_{\ell\neq 0} C_{\ell,k}.$$

Inserting this is eq (5) gives

$$\begin{aligned} C_{m,p} = \frac{\bar{\eta}q^2\omega_0^2 MN}{2\pi\omega_{s0}}\sum_k\frac{Z_k}{kM+s}m \\ \int_0^{\hat{\phi}}\frac{drJ_m[(pM+s)r]J_m[(kM+s)r]}{-i\Omega+im\omega_s(r)}F_0'(r) \\ \sum_{\ell\neq 0} C_{\ell,k} + drive. \end{aligned} \tag{6}$$

Summing over m gives the equation for the bunched beam transfer function

$$\begin{aligned} \rho_p - \frac{\bar{\eta}q^2\omega_0^2 MN}{2\pi\omega_{s0}}\sum_k\frac{Z_k}{kM+s}\rho_k\sum_m m \\ \int_0^{\hat{\phi}}\frac{drJ_m[(pM+s)r]J_m[(kM+s)r]}{-i\Omega+im\omega_s(r)}F_0'(r) \\ = drive. \end{aligned} \tag{7}$$

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The integral in equation (7) can be singular if $Im(\Omega) = 0$. For this case I consider $\Omega = Re(\Omega) + i0^+$, a small positive imaginary component corresponds to an adiabatic turn on of the driving force. Equation (7) is also valid for finite growth rates with $Im(\Omega) > 0$.

With $Im(\Omega) = 0^+$ it is possible that eq (7) will not have a solution for vanishing drive. This will be the case when the motion is damped or when the system is exponentially unstable, and this is the key to using it for predicting beam stability. In particular notice that (7) is of the form

$$[\mathbf{1} - N\mathbf{Z}(\Omega)]\mathbf{P} = \mathbf{D}, \tag{8}$$

where N is the number of particles per bunch, $\mathbf{P} = \rho_p$, $\mathbf{Z}$ is the rest of the dispersion matrix, and $\mathbf{D}$ is the drive. For small N, $\mathbf{P} \approx \mathbf{D}$ and the system is stable. As N grows the matrix $\mathbf{1} - N\mathbf{Z}(\Omega)$ changes until, for some $\Omega = \Omega_c$, $det(\mathbf{1} - N_t\mathbf{Z}(\Omega_c)) = 0$, where N_t is the threshold intensity for coherent frequency Ω_c. Therefore, to use (7) in a stability analysis one plots $det(\mathbf{1} - N\mathbf{Z}(\Omega)])$ on the complex plane as a function of Ω. Strictly speaking these plots should also be made for all values of N less than the value of interest. If none of the plots encircle the origin the matrix will have an inverse for the intensity of interest and the system will be stable. If the curve goes through the origin then the frequency Ω is an eigenfrequency. Similar work using just one matrix element was considered in [9].

To connect this formalism to the usual results note that retaining a single value of m in the summation of (7) and making the approximation $J_m(x) = (x/2)^m/m!$ leads to a matrix of rank one. The resulting coherent frequencies are similar to the handbook formulas [6].

APPLICATIONS

The theory in the previous section has been implemented in the fortran code NYQUIST. I assume the frequency shifts will be small so that only one synchrotron mode at a time needs to be considered. That is to say, the sum over m in (7) is replaced by a single value. The variation of synchrotron frequency with r is taken as $\omega_s(r) = \omega_{s0}[1 - (hr/4)^2]$ where h is the harmonic number. The infinite sum over k in (7) is truncated according to $|f_0(kM + s)| < f_{max}$. The impedance is modeled as a sum of resonators plus a constant, broad band Z/n. The dispersion integrals are treated numerically using a uniform grid in the action-like variable $I = (hr)^2$. For $Im(\Omega) =$ one takes

$$\frac{1}{0^+ - i\Omega + im\omega_{s0}I/16} = iPV\left\{\frac{1}{\Omega - m\omega_{s0}I/16}\right\} + \pi\delta\left(\Omega - m\omega_{s0}I/16\right) \tag{9}$$

By using an action grid $I_n = n\Delta I$ and a frequency grid $\Omega_k = k\Delta Im\omega_{s0}/16$, the delta function always corresponds to a lattice point and the principle value is just a sum with the single point corresponding to the resonant denominator removed. For $Im(\Omega) > 0$ we take $\Omega_k = i\epsilon + k\Delta Im\omega_{s0}/16$ to keep as much symmetry as possible. Also we take $\Delta Im\omega_{s0}/16 \lesssim \epsilon/5$ which corresponds to a 1.3% fractional error between summation and integration. The fractional error was estimated using (for $a = 5$)

$$\sum_{k=-\infty}^{\infty} \frac{a}{a^2 + k^2} = \sum_{k=-\infty}^{\infty} \pi e^{-a|k|} = \pi + \frac{2\pi}{e^a - 1}.$$

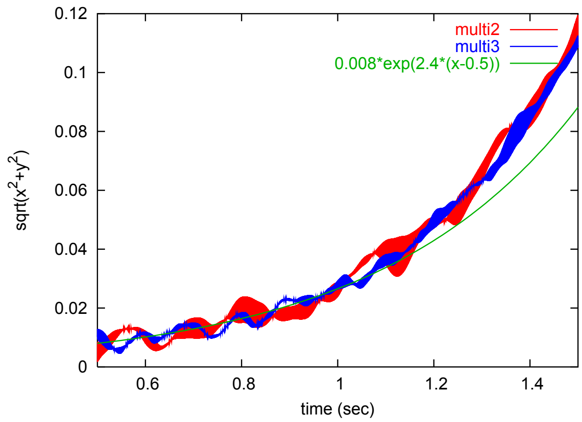

Figure 1: Comparison of multiparticle simulations and the threshold of NYQUIST.

The accuracy and convergence of NYQUIST has been benchmarked using simulations. Figure 1 shows the result of simulations of four bunches with 20,000 macroparticles each. The impedance consists of a broad band Z/n and a narrow band resonator. The third revolution hamonic of the beam current is plotted as a function of time. For the red curve labeled multi2 the intital phase space distribution of macroparticles was regular and the charge per macroparticle was adjusted to obtain the desired line density. For the blue curve labeled multi3 all macroparticles had the same charge and the initial phase space distribution was chosen to give the correct line density. The two curves agree rather well. The green curve shows an exponential with a $2.4\mathrm{s}^{-1}$ growth rate.

Figure 2 shows results from NYQUIST with a $2.4\mathrm{s}^{-1}$ growth rate for different values of f_{max}. All come close to the black dot at the origin but the agreement is not perfect. The equivalent length of the smoothing function was $\tau_s = 5$ ns. The corresponding upper frequency is $f_{max} = 1/2\tau_s = 100$ MHz so the the black curve is closest to the simulation. Figure 3 shows NYQUIST results near the origin for $f_{max} = 100$ MHz and a growth rate of $1.2\mathrm{s}^{-1}$ is found. This is a factor of 2 smaller than the simulation and it is possible that the smooth spectral cutoff used in the simulation behaves differently than truncating the matrix. Also note that the linear rf growth rate for this system is $4.8\mathrm{s}^{-1}$ so the effects of Landau damping are large. The code has been checked for internal consistency. In particular, the long wavelength limit has been checked and agrees quite well with the usual formulas [3, 4].

Now turn to the new 56 MHz cavity in RHIC. To get

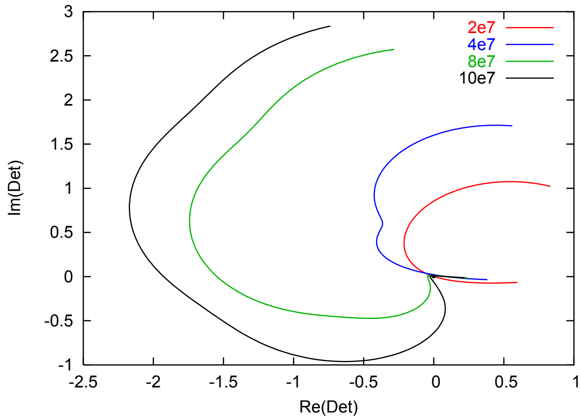

Figure 2: NYQUIST threshold curves versus upper frequency cutoff. $Im(\Omega) = 2.4\mathrm{s}^{-1}$.

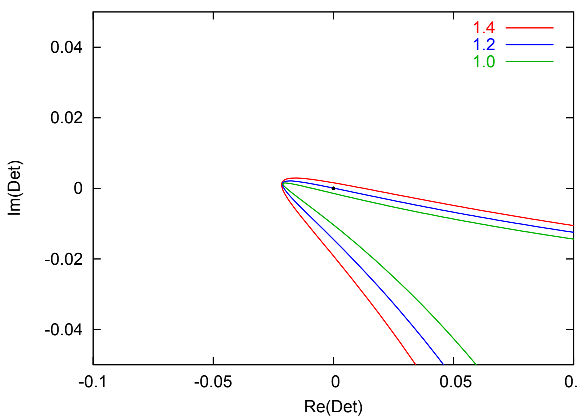

Figure 3: Curves for $f_{max} = 10^8 s^{-1}$ and different growth rates.

an idea of the worst case scenario, all the 56 MHz modes with a resonant frequency less than 600 MHz where shifted in frequency to drive the same coupled bunch mode with $s = 708$. In RHIC we operate with a 120 bunch symmetric fill pattern, minus an abort gap. With an upper frequency of 4 GHz this leads to a rather long execution time. The computations were sped up by assuming 720 bunches and dividing all narrow band impedances by 6. This led to a matrix size of 143×143 which executed in a reasonable time on a workstation. Results are shown in Fig. 4. The large broad band impedance with $Z/n \sim 3\Omega$ appears to dominate the threshold. This in turn leads to a strong bunch length dependence which has been appreciated for some time [11]. Finite matrices in NYQUIST follow from the approximation of a band limited impedance. Typically, other techniques employ a basis expansion which has different and possibly inferior [12] convergence properties.

CONCLUSIONS

Landau damping with high frequency impedance has been explored by examining the stability of the bunched beam transfer function. This technique allows for absolute tests of bunched beam stability, even in the limit of vanishingly small growth rate. If one can differentiate between short and long range wakes it is possible to reduce computation time by doing calculations for closely spaced bunches with a reduced long range wake. Finite matrices follow from the assumption of a band limited impedance as opposed to the truncation of a basis expansion.

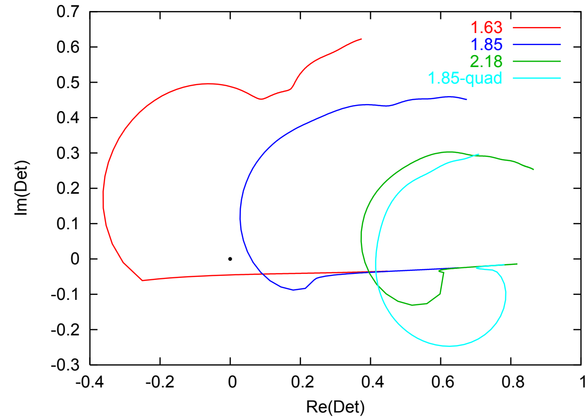

Figure 4: Plot of the determinant for $Im(\Omega) = 0$ and 1×10^{11} protons with 2.4 MV on the 56 MHz cavity. Bunch lengths with $\sigma_t = 1.6$ ns, 1.8 ns and 2.2 ns are shown. Also, the quadrupole $(m = 2)$ mode for the 1.8 ns mode is shown.

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