FOUR REGIMES OF THE IFR ION HOSE INSTABILITY*

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Abstract

An electron beam focused by an ion channel without a magnetic field, in the so-called ion focus regime (IFR), may be disrupted by the transverse ion hose instability. We describe the growth in four regimes.

INTRODUCTION

Propagation of an electron beam focused by a preformed ion channel may be disrupted by the ion hose instability [1–3]. In the rigid-beam model [1]

$$d^2b/dt^2 = -\omega_e^2(b-c), \quad d^2c/dt^2 = -\omega_i^2(c-b), \tag{1}$$

where $b(z,t)$ is the beam displacement, $c(z,t)$ is the channel displacement, z is axial location, t is time, $db/dt = \partial b/\partial t + v\partial b/\partial z$ and $dc/dt = \partial c/\partial t$ where v is the beam velocity, while ω_e and ω_i are the electron betatron frequency and the ion bounce frequency.

Equation (1) describes an absolute instability, while the ion hose instability is actually a convective instability where a growing disturbance moves downstream and towards the tail of the beam [2]. This may be remedied by considering distributions of betatron and bounce frequencies [2]. We model Cauchy (also called Lorentzian) distributions with half-widths α_e and α_i, whose frequency spreads give exponential decoherence of centroid oscillations approximated by linear damping [4]

$$\begin{aligned} d^2b/dt^2 &= -\omega_e^2(b-c) - 2\alpha_e\, db/dt, \\ d^2c/dt^2 &= -\omega_i^2(c-b) - 2\alpha_i\, dc/dt. \end{aligned} \tag{2}$$

Equation (2) also describes the electron hose instability of an electron beam that expels ions from uniform plasma [5, 6], the beam breakup (BBU) instability when the parameter $s_1 = 1$ [7], and the $e-p$ instability of a proton beam in a channel of electrons [8]. We model realistic damping with $\alpha_e/\omega_e = \alpha_i/\omega_i = 0.1$ [8].

DISPERSION RELATION

A disturbance that is dominated by a single frequency is described by the dispersion relation. For the ansatz $b(z,t) = b_0 e^{i(kz-\omega t)}$, $c(z,t) = c_0 e^{i(kz-\omega t)}$, we can solve Eq. (2) for $\Omega \equiv \omega - vk$ as a function of ω, or vice versa

$$\Omega = -i\alpha_e \pm \omega_e\left(1 - \frac{\alpha_e^2}{\omega_e^2} + \frac{\omega_i^2}{\omega^2 - \omega_i^2 + 2i\omega\alpha_i}\right)^{1/2},$$

$$\omega = -i\alpha_i \pm \omega_i\left(1 - \frac{\alpha_i^2}{\omega_i^2} + \frac{\omega_e^2}{\Omega^2 - \omega_e^2 + 2i\Omega\,\alpha_e}\right)^{1/2}. \tag{3}$$

For a growing disturbance dominated by $\omega = \omega_i$, solving for Ω gives the spatial growth rate and group velocity

$$\begin{aligned} -\mathrm{Im}(k) &\approx \frac{0.5\omega_e\omega_i^{1/2}}{v\alpha_i^{1/2}} - \frac{\alpha_e}{v}, \\ v_g &\approx \left[\frac{d\,\mathrm{Re}(k)}{d\,\mathrm{Re}(\omega)}\right]^{-1} \approx \frac{v}{1 + 0.25\omega_e\omega_i^{1/2}/\alpha_i^{3/2}}. \end{aligned} \tag{4}$$

For a growing disturbance dominated by $\Omega = \omega_e$, solving for ω gives the temporal growth rate and group velocity

$$\begin{aligned} \mathrm{Im}(\omega) &\approx \frac{0.5\omega_i\omega_e^{1/2}}{\alpha_e^{1/2}} - \alpha_i, \\ v_g &\approx \frac{d\,\mathrm{Re}(\omega)}{d\,\mathrm{Re}(k)} = \frac{d\,\mathrm{Re}(\omega)/d\Omega}{d\,\mathrm{Re}(k)/d\Omega} \approx \frac{v}{1 + \alpha_e^{3/2}/(0.25\omega_i\omega_e^{1/2})}. \end{aligned} \tag{5}$$

IMPULSE RESPONSE

In terms of betatron phase $Z \equiv \omega_e z/v$ and ion bounce phase $\xi \equiv \omega_i(t - z/v)$, Eq. (2) with an impulse becomes

$$\begin{aligned} \partial^2 b/\partial Z^2 &= -(b-c) - 2A_e\,\partial b/\partial Z + \delta(Z)\delta(\xi), \\ \partial^2 c/\partial \xi^2 &= -(c-b) - 2A_i\,\partial c/\partial \xi, \end{aligned} \tag{6}$$

where $A_e \equiv \alpha_e/\omega_e$ and $A_i \equiv \alpha_i/\omega_i$. For a beam whose head is at $\xi = 0$ that enters an ion channel at $Z = 0$, $b(Z,\xi)$ is the response to an impulsive force applied to the head of the beam at the entrance of the channel. For underdamped electron and ion oscillations with $A_e, A_i < 1$, the solution to Eq. (6) for an immobile ion channel with $c(Z,\xi) \equiv 0$ is [9]

$$b_0(Z,\xi) = \delta(\xi)e^{-A_e Z}\sin\left(Z\sqrt{1-A_e^2}\right)\Big/\sqrt{1-A_e^2}. \tag{7}$$

For mobile ions, the solution is the sum of Eq. (7) and [9]

$$\begin{aligned} \Delta b(Z,\xi) = e^{-A_e Z}e^{-A_i\xi}\sum_{k=1}^{\infty}\frac{\pi}{k!(k-1)!}\left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{k-1/2} \\ \times J_{k-1/2}\left(\xi\sqrt{1-A_i^2}\right)\left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{k+1/2} J_{k+1/2}\left(Z\sqrt{1-A_e^2}\right). \end{aligned} \tag{8}$$

A similar formula describes the mobile ions [9]. For a pulse length of $\xi/2\pi$ ion bounce periods, $\Delta b(Z,\xi)$ is the tail offset after propagating $Z/2\pi$ betatron wavelengths.

To approximate Eq. (8), we use the small and large argument approximations: $J_\nu(x) \approx [\Gamma(\nu+1)]^{-1}(x/2)^\nu$ for $x \ll \nu$, $J_\nu(x) \approx (2/\pi x)^{1/2}\cos(x - \nu\pi/2 - \pi/4)$ for $x \gg \nu$ [10], and an approximation of the gamma function: $\Gamma(n+1) \approx \sqrt{2\pi}\, n^{n+1/2}e^{-n}$ for $n \gg 1$.

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Short Propagation Distance of a Short Pulse

For $Z,\xi << 1$, the sum in Eq. (8) is dominated by the first term. Applying the small-argument approximation to the Bessel functions gives the non-oscillating result

$$\Delta b(Z,\xi) \approx \xi Z^3/6 . \quad (9)$$

Short Pulse

For $\xi << Z$, the pulse length (measured in ion bounce phase) is much shorter than the propagation distance (measured in electron betatron phase). For $\xi^{2/3}Z^{1/3} >> 1$, Eq. (8) is dominated by terms with $\xi << k << Z$, where the small-argument approximation applies to $J_{k-1/2}(\xi\sqrt{1-A_i^2})$ and the large-argument approximation applies to $J_{k+1/2}(Z\sqrt{1-A_e^2})$. Applying these approximations to all of the terms and using complex notation where the real part gives the physical disturbance, we have

$$\Delta b(Z,\xi) \approx \frac{\sqrt{\pi}\, e^{-A_e Z} e^{-A_i \xi} e^{iZ\sqrt{1-A_e^2}}}{\sqrt{1-A_e^2}} \times \sum_{k=1}^{\infty} \frac{1}{k!(k-1)!\Gamma(k+1/2)} \left(\frac{\xi}{2}\right)^{2k-1} \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{k} e^{-i\frac{\pi}{2}(k+1)} . \quad (10)$$

Using $k! \approx \sqrt{2\pi}\, k^{k+1/2} e^{-k}$, $(k-1)! = k!/k \approx \sqrt{2\pi}\, k^{k-1/2} e^{-k}$ and $\Gamma(k+1/2) \approx \sqrt{2\pi}\, k^k e^{-k}$ gives $k!(k-1)!\Gamma(k+1/2) \approx (2\pi/3^{3k})\Gamma(3k+1/2)$, which yields

$$\Delta b(Z,\xi) \approx \frac{\sqrt{3}\, e^{-A_e Z} e^{-A_i \xi} e^{iZ\sqrt{1-A_e^2}}}{2\sqrt{\pi}\sqrt{1-A_e^2}} \left(\frac{\xi}{2}\right)^{-2/3} \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/6} \times e^{-i\frac{7\pi}{12}} \sum_{k=1}^{\infty} \frac{1}{\Gamma(3k+1/2)} \left[3\left(\frac{\xi}{2}\right)^{2/3} \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/3} e^{-i\frac{\pi}{6}} \right]^{3k-1/2} . \quad (11)$$

The sum in Eq. (11) approximates every third term of the Taylor series for an exponential, so that

$$\Delta b(Z,\xi) \approx \frac{e^{-A_e Z} e^{-A_i \xi}}{2\sqrt{3\pi}\sqrt{1-A_e^2}} \left(\frac{\xi}{2}\right)^{-2/3} \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/6} \times e^{i\left(Z\sqrt{1-A_e^2} - G/\sqrt{3} - 7\pi/12\right)} e^{G} , \quad (12)$$

where for a short pulse

$$G = \left(3\sqrt{3}/4\right)\xi^{2/3}\left(Z\Big/\sqrt{1-A_e^2}\right)^{1/3} \quad (13)$$

is the exponential growth factor. This factor has been previously obtained when damping is neglected [2, 3], and corresponds to BBU growth type C of Ref. [7]. The additional exponential factor $-A_e z - A_i \xi$ gives damping.

For a given value of ξ (a slice of the beam), the envelope $|\Delta b(Z,\xi)|$ peaks where $Z \approx 0.285\xi / A_e^{3/2}$. The peak's velocity and temporal growth rate are $v/[1+\alpha_e^{3/2}/(0.285\omega_i\omega_e^{1/2})]$ and $0.57\omega_i\omega_e^{1/2}\alpha_e^{-1/2} - \alpha_i$, approximately given by Eq. (5) for $\Omega = \omega_e$.

Figure 1 displays the oscillating impulse response for a short pulse and its approximation by Eq. (12).

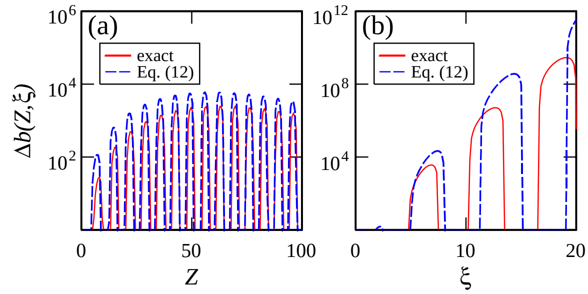

Figure 1: The impulse response function and the short-pulse approximation of Eq. (12), valid for $\xi << Z$. (a) $\xi = 2\pi = 6.28$. (b) $Z = 20\pi = 62.8$.

Long Pulse

For $Z << \xi$, the pulse length (measured in ion bounce phase) is much longer than the propagation distance (measured in betatron phase). For $Z^{2/3}\xi^{1/3} >> 1$, Eq. (8) is dominated by terms with $Z << k << \xi$, where the large- and small-argument approximations apply to $J_{k-1/2}(\xi\sqrt{1-A_i^2})$ and $J_{k+1/2}(Z\sqrt{1-A_e^2})$, respectively. Approximating all terms, using complex notation and $k!(k-1)!\Gamma(k+3/2) \approx (2\pi/3^{3k+1})\Gamma(3k+3/2)$, we have

$$\Delta b(Z,\xi) \approx \frac{\sqrt{3}\, e^{-A_e Z} e^{-A_i \xi} e^{i\xi\sqrt{1-A_i^2}}}{2\sqrt{\pi}\sqrt{1-A_i^2}} \left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{-7/6} \left(\frac{Z}{2}\right)^{2/3} \times e^{i\frac{\pi}{12}} \sum_{k=1}^{\infty} \frac{1}{\Gamma(3k+3/2)} \left[3\left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{1/3} \left(\frac{Z}{2}\right)^{2/3} e^{-i\frac{\pi}{6}} \right]^{3k+1/2} . \quad (14)$$

The sum in Eq. (14) approximates

$$\Delta b(Z,\xi) \approx \frac{e^{-A_e Z} e^{-A_i \xi}}{2\sqrt{3\pi}\sqrt{1-A_i^2}} \left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{-7/6} \left(\frac{Z}{2}\right)^{2/3} \times e^{i\left(\xi\sqrt{1-A_i^2} - G/\sqrt{3} + \pi/12\right)} e^{G} , \quad (15)$$

where for a long pulse

$$G = \left(3\sqrt{3}/4\right)\left(\xi\Big/\sqrt{1-A_i^2}\right)^{1/3} Z^{2/3} . \quad (16)$$

This factor has been previously obtained when damping is neglected [5, 6], giving BBU growth type A of Ref. [7].

For a given value of Z (axial location), $|\Delta b(Z,\xi)|$ peaks where $\xi \approx 0.285 Z / A_i^{3/2}$. The peak's velocity and spatial growth rate are $v/(1+.285\omega_e\omega_i^{1/2}/\alpha_i^{3/2})$ and $0.57\omega_e\omega_i^{1/2}\alpha_i^{-1/2}/v - \alpha_e/v$, approximated by Eq. (4) for $\omega = \omega_i$.

The impulse response for a long pulse and its approximation by Eq. (15) are shown in Fig. 2.

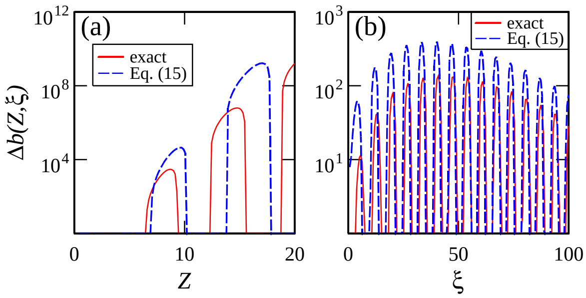

Figure 2: The impulse response function and the long-pulse approximation of Eq. (15), valid for $Z << \xi$. (a) $\xi = 20\pi = 62.8$. (b) $Z = 2\pi = 6.28$.

Medium Pulse Length

For $Z \sim \xi >> 1$, the propagation distance (measured in electron betatron phase) is comparable to the pulse length (measured in ion bounce phase). For $Z^{1/2}\xi^{1/2} >> 1$, Eq. (8) is dominated by terms with $k \sim \xi/2 \sim Z/2$, where the large-argument approximation applies to $J_{k-1/2}(\xi\sqrt{1-A_i^2})$ and $J_{k+1/2}(Z\sqrt{1-A_e^2})$. Approximating all terms, using complex notation and $k!(k-1)! \approx (\sqrt{2\pi}/2^{2k})\Gamma(2k+1/2)$ gives

$$\Delta b(Z,\xi) \approx \frac{e^{-A_e Z} e^{-A_i \xi} e^{i\xi\sqrt{1-A_i^2}}}{2\sqrt{\pi}\,\sqrt{1-A_e^2}\,\sqrt{1-A_i^2}} \left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{-3/4} \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/4}$$
$$\times \sum_{k=1}^{\infty} \frac{1}{\Gamma(2k+1/2)} \left[2\left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{1/2}\left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/2}\right]^{2k-1/2}$$
$$\times \left[e^{i\left(Z\sqrt{1-A_e^2}-3\pi/4\right)}\left(e^{-i\pi/2}\right)^{2k-1/2} + e^{-i\left(Z\sqrt{1-A_e^2}-\pi/2\right)}\right]. \tag{17}$$

The sum in Eq. (17) approximates

$$\Delta b(Z,\xi) \approx \frac{e^{-A_e Z} e^{-A_i \xi} e^{i\xi\sqrt{1-A_i^2}}}{4\sqrt{\pi}\,\sqrt{1-A_e^2}\,\sqrt{1-A_i^2}} \left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{-3/4} \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/4}$$
$$\times \left\{ \begin{array}{l} \exp\left[2\left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{1/2}\left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/2} e^{-i\pi/2}\right] e^{i\left(Z\sqrt{1-A_e^2}-3\pi/4\right)} \\ +\exp\left[2\left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{1/2}\left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/2}\right] e^{-i\left(Z\sqrt{1-A_e^2}-\pi/2\right)} \end{array} \right\}. \tag{18}$$

The growing term that dominates Eq. (18) is

$$\Delta b(Z,\xi) \approx \frac{e^{-A_e Z} e^{-A_i \xi}}{4\sqrt{\pi}\,\sqrt{1-A_e^2}\,\sqrt{1-A_i^2}} \left(\frac{\xi}{2\sqrt{1-A_i^2}}\right)^{-3/4}$$
$$\times \left(\frac{Z}{2\sqrt{1-A_e^2}}\right)^{1/4} e^{i\left(\xi\sqrt{1-A_i^2}-Z\sqrt{1-A_e^2}+\pi/2\right)} e^{G}, \tag{19}$$

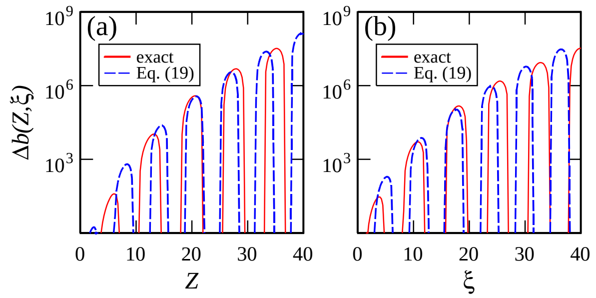

Figure 3: The impulse response function and the medium-pulse approximation of Eq. (19), valid for $Z \sim \xi >> 1$. (a) $\xi = 6\pi = 18.85$. (b) $Z = 6\pi = 18.85$.

where in the case of medium pulse length, the growth factor is

$$G = \left(\xi\Big/\sqrt{1-A_i^2}\right)^{1/2}\left(Z\Big/\sqrt{1-A_e^2}\right)^{1/2}. \tag{20}$$

This growth factor describes BBU growth when $s_2 \approx s_1 = 1$ in the notation of Ref. [7].

Figure 3 shows the impulse function in the medium-pulse-length regime and its approximation by Eq. (19). For $Z \sim \xi >> 1$, Eq. (19) provides a good approximation.

SUMMARY

The asymptotic growth of the IFR ion hose instability has been obtained in four regimes, including the well-known short-pulse and long-pulse regimes. We also found growth for a short pulse that is propagated for a short distance, and the asymptotic growth in the medium-pulse-length regime where the number of electron betatron oscillations during the beam's propagation is comparable to the number of ion oscillations during the beam's passage.

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