MATHEMATICA APPLICATION FOR METHODICAL IONIZATION COOLING CHANNEL DESIGN*

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Abstract

Existing codes for accelerator design (e.g. MAD) are not well suited for ionization cooling channels where particles exhibit strongly dissipative and nonlinear motion. A system of *Mathematica* programs was developed which allows to: 1) find periodic orbit and eigenvectors of the transfer matrix around it with account of (regular part of) ionization losses and feeddown effect from nonlinear fields; 2) compute emittance growth due to scattering and straggling, find equilibrium values (if exist); 3) analyze nonlinear effects such as dependence of tunes and damping rates on the amplitudes, resonance excitation; 4) perform tracking as the ultimate test of design. Underlying theory and application to helical FOFO snake are presented.

INTRODUCTION

Development of a 6D ionization cooling channel (ICC) which would allow to achieve high phase space density of muons is fundamental for realization of a high luminosity Muon Collider [1]. For successful design of such a channel a trade-off must be found between contradictory requirements of particle stability at large amplitudes and strong focusing needed for low equilibrium emittance.

The widely used tools – ICOOL [2] and G4BL [3] – rely mostly on massive particle tracking and do not provide that level of insight which is characteristic to standard codes like Methodical Accelerator Design (MAD) [4]. Unfortunately such standard codes are not particularly useful for cooling channel design since they are based on Hamiltonian dynamics in the field of hard-edge magnetic elements (and RF cavities).

On the contrary, magnetic elements to be used in ICC (mainly solenoids) have quite large aperture to length ratio so that there is significant overlap of the fields generated by different elements. And, of course, the primary goal of ICC is to render the muon dynamics non-Hamiltonian.

Therefore there is a need for a MAD-like code which would include:

- long-range fields of tilted and displaced off-axis magnetic elements,
- fully coupled 6D optics functions calculation in presence of strong damping,
- analysis of higher order effects on beam dynamics (e.g. damping decrement dependence on the amplitudes of oscillations)

Here we describe an attempt to build such a code which we call – in the spirit of MAD – the Methodical Ionization Cooling Channel Design (MICCD). For now it is implemented a collection of *Mathematica* [5] notebooks and is limited to the case of a straight reference orbit (channel axis). As a consequence, the real orbit should not deviate from the axis by more than ~50% of the channel inner radius.

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BASIC EQUATIONS

Let us choose the path length along the reference orbit (z-coordinate here) as the independent variable and dynamical variables in the form

$$\underline{u} = \{x, p_x, y, p_y, z - c\beta_0 t, \delta_p\} \tag{1}$$

with $p_{x,y}$ being canonical momenta normalized by the reference value p_0=$mc\beta_0\gamma_0$, and

$$\delta_p = (\gamma - \gamma_0) / \beta_0^2 \gamma_0 \,. \tag{2}$$

Equations of motion can be written in the form

$$\underline{u}' \equiv \frac{d}{dz}\underline{u} = \underline{F} = \mathrm{S}\cdot\frac{\partial}{\partial \underline{u}}\mathcal{H} + \underline{F}^{(\mathrm{ion})}, \tag{3}$$

where S is 6×6 symplectic unit matrix

$$\mathrm{S} = \mathrm{S}_2 \oplus \mathrm{S}_2 \oplus \mathrm{S}_2, \;\; \mathrm{S}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4}$$

$\mathcal{H}$ is normalized by p_0 Hamiltonian

$$\mathcal{H} = \delta_p - \kappa A_z - [1 + 2\delta_p + \beta_0^2\delta_p^2 - (p_\perp - \kappa A_\perp)^2]^{1/2} \tag{5}$$

with κ=$1/B\rho$ and $\boldsymbol{A}$ being the electromagnetic field vector-potential, $\underline{F}^{(\mathrm{ion})} = \underline{F}^{(\mathrm{reg})} + \underline{F}^{(\mathrm{stoch})}$ is the ionization recoil force. The regular part can be approximated as

$$\underline{F}^{(reg)} = -\frac{1 + \alpha_p (\ln \beta^2\gamma^2 - \beta^2)}{\beta_z \beta \beta_0^2 \gamma_0 L_{es}} \{0, \frac{\beta_0\beta_x}{\beta^2}, 0, \frac{\beta_0\beta_y}{\beta^2}, 0, 1\} \tag{6}$$

with just two parameters. For liquid H_2 L_{es}=4.57m, α_p=0.0914, for Li L_{es}=1.51m, α_p=0.0985.

The stochastic part is due to multiple scattering

$$\underline{F}^{(scat)} = [\frac{d}{dz}\theta_{sc}^2]^{1/2} \xi_1(z) \{0, \cos\chi, 0, \sin\chi, 0, 0\} \tag{7}$$

and straggling

$$F_6^{(strag)} = \frac{1}{\beta_0^2\gamma_0}[\frac{d}{dz}\Delta\gamma^2]^{1/2} \xi_2(z), \;\; F_{i\neq 6}^{(strag)} = 0 \tag{8}$$

In eqs.(7, 8) $\xi_{1,2}$ are uncorrelated stochastic variables:

$$\langle \xi_i(z) \xi_k(z') \rangle = \delta_{ik} \delta(z - z') \,, \tag{9}$$

χ is random angle. Corrections due to slope of the trajectory were omitted in eqs.(7, 8) for clarity but taken into account in the program.

In order to reduce computation time the field vector-potential is expanded in power series up to the 5th order in transverse coordinates

$$A = \sum_{j+k\le 5} A_{jk}(z)\frac{x^j y^k}{j!k!} \qquad (10)$$

For a thin coil this provides sufficient precision at distances from its axis up to 50% of the radius.

Solenoids are treated as arrays of thin coils using simple formulas with full elliptic integrals for high order derivatives.

The most difficult is treatment of wedge absorbers since z-coordinate of the entrance and exit points depends on the transverse position of the particle. We circumvent this difficulty by considering the wedge a thin element with local surface density proportional to its actual width.

ORBIT & LINEAR OPTICS

The cooling channels are necessarily periodic or quasi-periodic due to relatively small cooling rates. Therefore it is instructive to find periodic orbit and optics functions in absence of stochastic forces (7, 8).

Presenting the particle motion as superposition of motion along the periodic orbit and oscillations around it

$$\underline{u} = \underline{u}_{po} + \underline{\eta} \qquad (11)$$

and expanding the r.h.s of eq.(3) in $\underline{\eta}$ we obtain linear matrix equation whose solution can be presented with transfer matrix

$$\underline{\eta}(z) = \mathrm{M}(z,0)\underline{\eta}(0) \qquad (12)$$

The transfer matrix has a system of eigenvectors and eigenvalues which in the case of linearly stable motion form complex conjugate pairs

$$M\underline{v}_m = \lambda_m \underline{v}_m, \qquad (13)$$
$$\lambda_m = \exp[2\pi(iQ_m - \gamma_m)],\ \ \lambda_{m+1} = \lambda_m^*,\ \ m = 1, 3, 5$$

Decrements γ_k describe oscillation damping due to regular part of ionization losses. The action variable decrease over one period as

$$J_m(L) = \exp(-4\pi\gamma_m)J_m(0) \qquad (14)$$

Eigenvectors can be normalized as in the Hamiltonian case (but there is no simple orthogonality condition)

$$\underline{v}_m^+ \mathrm{S}\, \underline{v}_m = i \qquad (15)$$

Using eigenvectors as columns we can built matrix V, $V_{ik}=(\underline{v}_k)_i$, rendering the normal mode expansion in the form

$$\underline{\eta} = \mathrm{V}\underline{a} \qquad (16)$$

Generally the periodic orbit is not known in advance and must be found by an iterative process. With transfer matrix and its eigenvectors calculated at step n we can found next approximation for the periodic orbit

$$\underline{u}_{po}^{(n+1)} = \underline{u}_{po}^{(n)} + \mathrm{V}\underline{a}, \qquad (17)$$
$$a_k = -\frac{1}{1-e^{-2\pi(iQ_k-\gamma_k)}}\int_0^L \{\mathrm{V}^{-1}[\underline{F}(\underline{u}_{po}^{(n)}) - \underline{u}_{po}^{\prime(n)}]\}_k\, dz$$

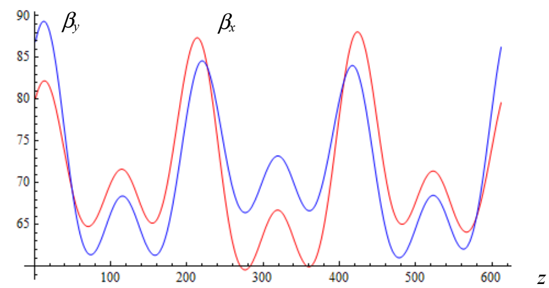

Figure 1: β-wave excited in “helical snake” due to difference in currents in solenoids of opposite polarity

Linear Optics Functions

With normalization (15) the Mais-Ripken β-functions [7] for mode μ can be computed as

$$\beta_{x\mu} = \frac{1}{2}e^{4\pi\gamma_{2\mu-1}z/L}\,|(\underline{v}_{2\mu-1})_1|^2, \ldots,\ \ \mu = I, II, III \qquad (18)$$

Since emittances of the transverse modes should not differ much, the beam sizes are determined by the sums

$$\beta_x = \beta_{xI} + \beta_{xII},\ \ \beta_y = \beta_{yI} + \beta_{yII}. \qquad (19)$$

There is no generally accepted definition of dispersion function for the case of fast longitudinal oscillations. We use the following recipe: let the phase of the 3rd mode oscillations be such that its projection on longitudinal coordinate u_5=2Re($V_{55}a_5$)=0. Then for dispersion we take the ratio of the 3rd mode projections on the transverse coordinate and momentum:

$$D_x = \frac{\mathrm{Im}(V_{56}V_{15})}{\mathrm{Im}(V_{56}V_{65})},\quad D_y = \frac{\mathrm{Im}(V_{56}V_{35})}{\mathrm{Im}(V_{56}V_{65})} \qquad (20)$$

Damping Partition & Equilibrium Emittances

Knowledge of the transfer matrix eigensystem allows to analyze the effect of linear perturbations, e.g. the change in cooling decrements produced by a wedge absorber depending on its position.

Let the transfer matrix through the wedge be I+W and M_1 be the transfer matrix from the origin to its position. With the help of the m-th mode projector $\mathrm{P}_m \underline{v}_k = \delta_{mk}\underline{v}_m$, and condition (15) we find for perturbation of the m-th eigenvalue

$$\lambda_m^{(1)} = -i\underline{v}_m^+ \mathrm{S}\mathrm{P}_m \mathrm{M}\mathrm{M}_1^{-1}\mathrm{W}\mathrm{M}_1\, \underline{v}_m \qquad (21)$$

Considering the stochastic component of the force $\underline{F}^{(\mathrm{stoch})} = \underline{f}\cdot\xi(z)$ as perturbation we can find the statistical average of the action variable increment over one period

$$\langle \Delta J_m \rangle = \int_0^L |(V^{-1}\underline{f})_m|^2\, dz \qquad (22)$$

and, taking into account damping (14), the r.m.s. emittance of the normal mode

$$\varepsilon_m \equiv \langle J_m \rangle = \frac{\langle \Delta J_m \rangle}{1-\exp(-4\pi\gamma_m)} \qquad (23)$$

Let us refer to [6] for an example of cooling channel – "helical FOFO snake" – studied using the described here approach. One period of the snake consists of 6 tilted alternating solenoids and two-cell RF cavities between them. One of the difficulties encountered with that design was unequal damping rates of the transverse modes. It was possible to equalize the rates with small spread in currents in "+" and "–" solenoids (Table 1). However, this excited large β-wave (Fig. 1) so that there was no reduction in equilibrium emittance.

Table 1: Complex Tunes vs. Solenoid Currents Ratio

$I+/I–$	$Q_I+i\gamma_I$	$Q_{II}+i\gamma_{II}$	$Q_{III}+i\gamma_{III}$
1	1.239+0.012i	1.279+0.007i	0.181+0.002i
1.016	1.212+0.010i	1.301+0.011i	0.196+.0003i

We considered another possibility to equalize the damping rates by putting Be wedges of two orientations at different points over the snake period. As Fig. 2 shows in all cases the damping rate of the second mode was further reduced.

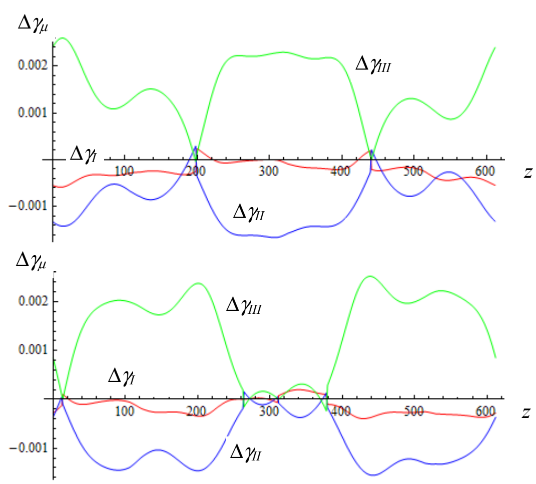

Figure 2: Change in cooling decrements vs. position of horizontal (top) and vertical (bottom) Be wedge of full angle 0.1rad.

NONLINEAR DYNAMICS

The linear normal mode analysis lays foundation for nonlinear perturbation theory. We use the extension of Deprit's algorithm for nonlinear non-Hamiltonian systems as outlined in [8] for the case of radiative particle dynamics. Application of this algorithm gives transformation from nonlinear "true" invariants c_m to linear normal forms a_m connected to dynamic variables by eq.(16) as a power series expansion

$$a_m = c_m + \sum_{jk} G^{(1)}_{mjk} c_j c_k + \sum_{jkl} G^{(2)}_{mjkl} c_j c_k c_l + \ldots \qquad (24)$$

As a by-product it gives nonlinear corrections to eigenvalues (i.e. tunes and damping rates) and resonance driving terms.

Table 2: Complex Detuning Coefficients

	$\partial(Q_I+i\gamma_I)$	$\partial(Q_{II}+i\gamma_{II})$	$\partial(Q_{III}+i\gamma_{III})$
$/\partial J_I$	-.0169+.0004i	-.0159-.0031i	-.0129+.0020i
$/\partial J_{II}$	-.0168-.0026i	-.0119+.0009i	-.0191+.0008i
$/\partial J_{III}$	-.0132-.0010i	-.0189-.0024i	-.0071+.0017i

Tune derivatives w.r.t. action variables of the normal modes calculated for the reference "snake" design are given in Table 2. The most troublesome is fast decrease in the 2nd mode damping rate with amplitudes of the 1st and 3rd modes. However, this may not lead to particle loss if these mode themselves are damped.

The nonlinear normal mode analysis provides insight into the role of different sources of perturbation and helps in the search for optimum parameters of the channel. However, the ultimate test of the design is tracking which reveals the concerted effect of all sources of perturbation. At the current stage of MICCD development only tracking without stochastic effects is implemented.

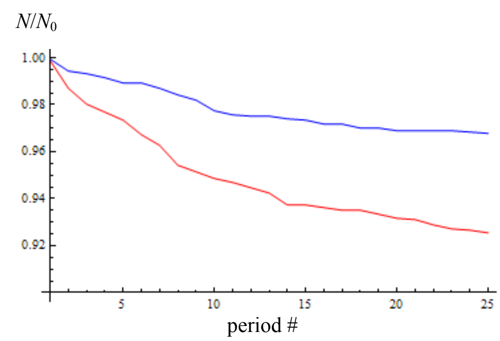

Figure 3: Transmission with (red) and without (blue) damping rates equalization.

Fig. 3 shows the effect of damping rates equalization as shown in Table 1 on transmission over 25 periods of the FOFO snake (153m). One can see that the rate equalization not only fails to reduce equilibrium emittance but also compromises transmission.

The linear and nonlinear normal mode analysis undoubtedly will become a powerful tool in the design of cooling channels. The *Mathematica* prototype of MICCD has already proved its usefulness in the study of helical FOFO snake [6].

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