MATRIX SOLUTION OF COUPLING IMPEDANCE IN MULTI-LAYER CIRCULAR CYLINDRICAL STRUCTURES*

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Abstract

Continuing interest in computing the coupling impedance of cylindrical multi-layer beam tubes led to several recent publications. A novel matrix method is here presented in which radial wave propagation is treated in analogy to longitudinal transmission lines. Starting from the Maxwell equations the solutions for monopole electromagnetic fields are in each layer described by a 2×2 matrix. Assuming isotropic material properties within one layer, the radially transverse field components at the inner boundary of a layer are uniquely determined by matrix transfer of the field components at its outer boundary. By imposing power flow constraints on the matrix, field matching between layers is enforced and replaced by matrix multiplication. The coupling impedance of a stainless steel beam tube defined by a matrix is given as a representative demonstration.

INTRODUCTION

The well known longitudinal resistive wall impedance was derived by Neil and Sessler [1] for an infinitely thick beam tube. Zotter [2] in a seminal paper gives the impedance of beam tubes from various materials but with finite wall thickness. The solutions for multilayer structures are typically based on an algorithm involving field matching at the boundary layers and sequential matching of radial wave impedances [3]. Although this method in principle allows many layers, the numerical implementation becomes increasingly complex and can be simplified by the use of a matrix method.

In the course of the study of the coated ceramic beam tube in the RHIC injection kicker, this author noticed the analogy of radial with longitudinal wave transmission and conceived a novel method in which the sequential wave impedance matching is replaced with multiplication of appropriate matrices relating the electric and magnetic field components in each layer [4]. Independently and without reference to transmission lines, Lambertson applied matrices to a double metal layer [5]. Using the theory of transmission lines, Vos derived expressions for the longitudinal impedance of multi-layer vacuum chambers, but without using matrices [6]. Recently, similar concepts were presented as field transformation matrix formalism [7].

The matrix solution presented here is characterized by a strict separation of the impedance contribution from the space charge and from the surface impedance at the beam tube wall. The beam tube impedance is obtained as solution of the homogeneous vector wave equation and is independent of the driving current. Continuity of radial power flow in the absence of the driving current is assured by appropriate constraints on the matrix describing each layer. Obviously, the transfer of field components across the matrix implies the transfer of impedances.

FIELD PRESENTATION

The electro magnetic field and its associated coupling impedance in a longitudinally uniform axially symmetric circular beam tube excited by a time harmonic current, are conveniently derived from the wave equation for the longitudinal electric field component, E_z, in natural units ($c=1$, $\mu_0=1, \varepsilon_0=1$, but $Z_0=120\pi\ \Omega$ if shown) [8]

$$\frac{d^2E_z}{dr^2}+\frac{dE_z}{rdr}+\left(\mu\varepsilon-\beta^{-2}\right)k^2E_z=\frac{j}{\varepsilon_S}\left(\mu\varepsilon-\beta^{-2}\right)ki_z \quad (1)$$

A logarithmic divergence in the space charge impedance is avoided using a tubular beam with radius, a, current, I, traveling in z-direction with velocity βc, and the current density

$$i_z=\frac{I}{2\pi a}\delta(r-a)e^{-j(\beta^{-1}kz-\omega t)} \quad (2)$$

where $k=\omega/c$ and the time dependence $e^{j\omega t}$ omitted. In the circular symmetric geometry considered here, the monopole electric field in any cylinder region is formed as linear combinations of cylinder functions, written in terms of Bessel functions or for this paper in terms of modified Bessel functions with argument (κr). The radial propagation constant, $\kappa^2=(\beta^{-2}-\mu\varepsilon_S)k^2$ is determined by the material parameters permeability $\mu=\mu'-j\mu''$ and permittivity plus conductivity $\varepsilon_S=\varepsilon-j\sigma/k$. The e.m. fields are here written with α as free parameter,

$$E_z(r)=K_0(\kappa r)+\alpha I_0(\kappa r)$$

$$H_\Theta(r)=j\frac{\varepsilon_S k}{\kappa}\frac{dE_z(\kappa r)}{d(\kappa r)}=j\frac{\varepsilon_S k}{\kappa}\{-K_1(\kappa r)+\alpha I_1(\kappa r)\} \quad (3)$$

The radial wave impedance is position dependent and given as

$$Z_\kappa=-\frac{E_z}{H_\Theta}=j\frac{\kappa}{\varepsilon_S k}\frac{K_0(\kappa r)}{K_1(\kappa r)}\sim j\frac{\kappa}{\varepsilon_S k} \quad (4)$$

In a single layer infinite beam tube, inner radius b, the coupling impedance is determined by the "wall impedance" $Z(b)=Z_\kappa$.

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BEAM TUBE IMPEDANCE

The foundation for the present study is laid by first deriving the expression for the longitudinal coupling impedance seen by an axial beam in a beam tube with known wall or "surface" impedance, $Z(b)$. The beam tube properties are fully defined at the inner tube radius by its wall impedance which can be found independently of the beam even in the case of a layered tube (of course assuming only linear materials). It is important to note that the total coupling impedance consists of two parts, the space charge plus a separate contribution from the beam tube. The space charge is found by considering the beam tube as a perfect conductor. Space charge and wall impedance are energy dependent. Although this paper treats the dependence rigorously, the final results are given for an extreme relativistic and filamentary beam.

The e.m. fields in the beam tube generated by the tubular current, with the common harmonic factor omitted and noting that in contrast to many papers $k=\omega$, are found to be inside the current tube

$$E_{zi}(\eta r) = A I_0(\eta r)$$
$$H_{\Theta i}(\eta r) = jA\beta\gamma I_1(\eta r) \qquad (5)$$

and outside of the current tube

$$E_{zo}(\eta r) = A\frac{I_0(\eta a)\left[K_0(\eta r)+\alpha I_0(\eta r)\right]}{K_0(\eta a)+\alpha I_0(\eta a)},$$
$$H_{\Theta o}(\eta r) = jA\beta\gamma\frac{I_0(\eta a)\left[-K_1(\eta r)+\alpha I_1(\eta r)\right]}{K_0(\eta a)+\alpha I_0(\eta a)} \qquad (6)$$

with $\eta = k/\beta\gamma$, β the relativistic velocity factor, γ the relativistic mass factor, and A and α free coefficients yet to be determined. Matching of $E_z(r)$ is built into the expressions and imposing Ampere's law at $r=a$ leads to

$$\mathrm{A} = j\frac{k}{2\pi a(\beta\gamma)^2}\left[K_0(\eta a)+\alpha I_0(\eta a)\right] \qquad (7)$$

Matching the fields to the wall impedance value yields

$$\alpha = \frac{j\beta\gamma Z(b)K_1(\eta b)-K_0(\eta b)}{I_0(\eta b)+j\beta\gamma Z(b)I_1(\eta\kappa b)} \qquad (8)$$

Finally, the longitudinal coupling impedance per unit length follows as Equ. 9

$$Z_{\|} = -\frac{E_z(a)}{I} =$$
$$-j\frac{kI_0(\eta a)}{2\pi(\beta\gamma)^2}\left\{K_0(\eta a)-\frac{I_0(\eta a)\left[K_0(\eta b)-jZ(b)\beta\gamma K_1(\eta b)\right]}{I_0(\eta b)+jZ(b)\beta\gamma I_1(\eta b)}\right\}$$

The total result can be separated into the space charge plus resistive wall impedance, Equ.10,

$$Z_{\|} = -j\frac{k}{2\pi(\beta\gamma)^2}\frac{I_0(\eta a)}{I_0(\eta b)}\left[I_0(\eta b)K_0(\eta a)-I_0(\eta a)K_0(\eta b)\right]+$$
$$+\frac{Z(b)}{2\pi b}\frac{I_0^2(\eta a)}{I_0^2(\eta b)+jZ(b)\beta\gamma I_0(\eta b)I_1(\eta b)}$$

The total impedance is simplified for the relativistic limit $\eta\to\infty$, and one step further for a filamentary beam into the expression in Equ. 11

$$\lim_{\gamma\to\infty} Z_{\|} \approx -j\frac{k}{2\pi(\beta\gamma)^2}\ln\left(\frac{b}{a}\right)$$
$$+\frac{Z(b)}{2\pi b}\left(\frac{1}{1+jkbZ(b)/2}\right)-j\frac{k}{8\pi(\beta\gamma)^2}\left\{\frac{8kbZ(b)+j3(kbZ(b))^2}{(1+jkbZ(b)/2)^2}\right\}$$

which is a more general solution than Chao's "Exercise 2.1" [9] but reduces in the low frequency limit to his

$$\lim_{\omega\to\infty} Z_{\|} \approx -\frac{Z(b)}{2\pi b} \qquad (12)$$

MATRIX METHOD

The e.m. fields and the coupling impedance of a beam tube with infinite radial extent is discussed above. The fields in a tube with finite radial thickness must satisfy additional boundary conditions at the outer radius, but are conveniently described by a matrix relating the field components at a radius within the layer to those at the outer radius, written here in the general form (and in natural units to stream line the notation) of Equ. (13)

$$\begin{pmatrix}E_z(r)\\ H_\Theta(r)\end{pmatrix} = M(r,r_0) = \begin{bmatrix}m_{ee}(r,r_0) & m_{eh}(r,r_0)\\ m_{he}(r,r_0) & m_{hh}(r,r_0)\end{bmatrix}\begin{pmatrix}E_z(r_0)\\ H_\Theta(r_0)\end{pmatrix}$$

The matrix elements must satisfy certain constraints to achieve power flow in addition to field component matching, implying that $\det M(r,r_0) = r_0/r$, and that at the reference radius, $r=r_0$, the matrix reads as

$$M(r_0,r_0) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \qquad (14)$$

The coupling impedance of a multi-layer structure is obtained by properly matching of the tangential field components at the cylinder boundaries. In full analogy to the treatment of longitudinal transmission lines the matching is best done with radial transfer matrices for each layer. In this method, the fields at the inner most layer, $r=b$ are determined via an overall matrix by the wave impedance of the outermost layer at r_o. The matrix of a sequence of radially spaced cylinders is found as the sequential product of all individual matrices, starting from the most inward radius, b, to the outermost radius r_o, [4]

$$M(b,r_o) = M^I(b,r_{0I})M^{II}(r_{II},r_{0II})..M^X(r_X,r_o) \qquad (15)$$

The rigorous expressions for the matrix elements in a layer with radial wave number κ and ε_S can be written in terms of modified Bessel functions as,

$$m_{ee}(r,r_0) = \kappa r_0\left[K_0(\kappa r)I_1(\kappa r_0)+I_0(\kappa r)K_1(\kappa r_0)\right]$$
$$m_{eh}(r,r_0) = j\frac{\kappa^2 r_0}{\varepsilon_S k}\left[K_0(\kappa r)I_0(\kappa r_0)-I_0(\kappa r)K_0(\kappa r_0)\right]$$
$$m_{he}(r,r_0) = -j\varepsilon_S k r_0\left[K_1(\kappa r)I_1(\kappa r_0)-I_1(\kappa r)K_1(\kappa r_0)\right]$$
$$m_{hh}(r,r_0) = \kappa r_0\left[K_1(\kappa r)I_0(\kappa r_0)+I_1(\kappa r)K_0(\kappa r_0)\right] \qquad (16)$$

The matrix multiplication is readily performed by the Wolfram-Mathematica program and does not need explicit expressions for the elements in the overall matrix. However, approximate matrix expressions are instructive:

Matrix of Air (Vacuum) between Two Layers

Following the usual practice of ignoring the small, non-zero electric susceptibility, $(\varepsilon'-1)\approx 6\times10^{-4}$ for the sake of simplicity, air is treated as vacuum. A practical matrix (and the extreme relativistic approximation, $\kappa\to\eta\simeq k/\beta\gamma\to 0$) for vacuum between two layers, r and r_0, follows as

$$m_{ee}(r,r_0)\simeq 1+\frac{r_0^2k^2}{4(\beta\gamma)^2}\left(\frac{r^2}{r_0^2}-1+2\ln\frac{r_0}{r}\right)\to 1$$

$$m_{eh}(r,r_0)\simeq j\frac{kr_0}{(\beta\gamma)^2}\ln\frac{r_0}{r}\to 0$$

$$m_{he}(r,r_0)=j\frac{kr}{2(\beta\gamma)^2}\left[\frac{r^2}{r_0^2}-1\right]\to 0$$

$$m_{hh}(r,r_0)\simeq\frac{r_0}{r}\left\{1+\frac{r^2k^2}{4(\beta\gamma)^2}\left(\frac{r_0^2}{r^2}-1+2\ln\frac{r}{r_0}\right)\right\}\to\frac{r_0}{r}\quad(17)$$

Matrix of high-conductivity metal

The high conductivity of a metal allows the simplification of κ to $\chi=\sqrt{j\mu\sigma k}$ and ε_S to $\varepsilon_\sigma\approx -j\sigma/k$ resulting in the well known approximate expressions

$$m_{ee}(r,r_0)=\sqrt{r_0/r}\cosh\chi(r_0-r)$$

$$m_{eh}(r,r_0)=-\sqrt{r_0/r}\,\frac{\chi}{\sigma}\sinh\chi(r_0-r)$$

$$m_{he}(r,r_0)=-\sqrt{r_0/r}\,\frac{\sigma}{\chi}\sinh\chi(r_0-r)$$

$$m_{hh}(r,r_0)=\sqrt{r_0/r}\cosh\chi(r_0-r)\quad(18)$$

IMPEDANCE MAPPING

The e.m. field pair at the outer radius of a layer is "mapped" by the matrix to the inside and thereby also the impedance. This can be generalized to the case of a multi-layer beam tube.

The coupling impedance seen by the beam is determined according to Equ. 11 from the wall or surface impedance $Z(b)$ at the inner beam tube radius, which in turn is found from the "mapped" wave impedance, $Z(r_o)$ of the infinitely extended layer beyond the outer radius of the beam tube. The wall impedance is in terms of the field pair given by

$$Z(b)=-\frac{E_z(r_o)}{H_\Theta(r_o)}=\frac{-E_z(r_o)M_{ee}(b,r_o)-H_\Theta(r_o)M_{eh}(b,r_o)}{E_z(r_o)M_{he}(b,r_o)+H_\Theta(r_o)M_{hh}(b,r_o)}\quad(19)$$

with the matrix elements found by multiplication according to Equ. 15 and in terms of impedances

$$Z(b)=\frac{Z(r_o)M_{ee}(b,r_o)-M_{eh}(b,r_o)}{M_{hh}(b,r_o)-Z(r_o)M_{he}(b,r_o)}\quad(20)$$

Although strictly speaking, the infinite terminal layer must be air, in practical terms, one can take a perfect conductor, or an infinitely thick metal cylinder with the wave impedances, $Z(r_o)=0$, and $Z(r_o)=\sqrt{j\mu k/\sigma}$, respectively. Note that the matrix solution covers these and more general situations equally well without change to the matrices.

The case of vacuum has been treated in different ways. Vacuum does not force the fields in the last layer, and should be treated as an antenna radiation problem. Most publications use $Z_\eta\sim j/\beta\eta$ which for the last material layer leads in the extreme relativistic limit to the implausible $E_z(r_o)\to 0$. This author is exploring the use of eigenvectors of $M(b,r_o)M(b,r_o)^{-1}$. Vos introduced the heuristic concept of an inductive by-pass [10], in which the free space impedance is used with $Z(r_o)=1$ for

$$Z(b)=Z_0\frac{M_{ee}(b,r_o)-M_{eh}(b,r_o)}{M_{hh}(b,r_o)-M_{he}(b,r_o)}\quad(21)$$

CONCLUSION

The matrix method [4], resuscitated here, to compute the coupling impedances of single and multi-layer beam tubes is demonstrated with an inductive by-pass in Fig. 1.

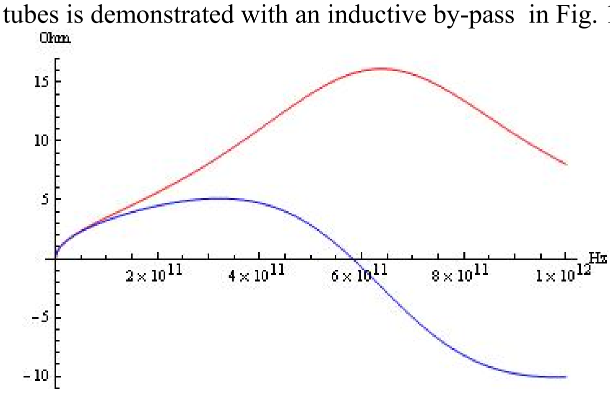

Figure 1: Real (red) and imaginary (blue) impedance per unit length of a straight metal tube with $b = 23.5$ mm, wall thickness 2 mm, and $\sigma = 1.5\times10^6/\Omega$m [7]

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