MODEL INDEPENDENT ANALYSIS WITH COUPLED BEAM MOTION*

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Abstract

This paper describes the results of measurements compared with the analysis of errors for a method of determining accelerator Twiss and coupling parameters from the singular value decomposition (SVD) of beam position monitor (BPM) data, taken on a turn-by-turn basis for a storage ring in fully coupled transverse beam coordinates. Using the transversely coupled-coordinate formalism described by Billing et al[1], the measurement technique expands on the work of Wang et al[2], which describes the SVD of the same data under the assumptions of no transverse coupling of the beam parameters. This particular method of data analysis requires a set of BPM measurements, taken when the beam is resonantly excited in each of its two dipole, betatron normal-modes of oscillation.

OUTLINE OF BASIC FORMALISM

In particle accelerators, measurements using BPM's have enabled the observation of properties such as Twiss and coupling parameters inferred from the position of the beam during the resonant excitation of both dipole modes of the beam[3]. The multi-turn position measurements are harmonically analyzed to calculate the betatron phase advance and some properties of the local coupling. A second method, described by Wang et al[2] for the case when the accelerator has no transverse coupling, excites a beam by an injection kicker and uses a SVD model independent analysis (MIA). This method suffers as the damping of the dipole oscillation limits the number of turns having sufficient signal amplitude. Both methods are relatively fast, but they require the excitation amplitude be known in order to determine the magnitude of the β functions and coupling parameters.

These two methods may be combined by using a constant amplitude resonant excitation together with the turn-by-turn BPM measurements and a SVD model independent analysis. If additional information about the known transport between of some subset of the BPMs is utilized, the excitation amplitudes may be calculated. This allows the computation of all of the single turn transport matrix elements for each of this subset of BPMs and other accelerator parameters at all of the BPMs, which would not otherwise have been determined.

The basic measurement and MIA may be summarized as follows: Resonantly excite the beam in first one and then the other dipole-normal mode of oscillation, while recording the BPM responses as a function of time. Create a position-history matrix of the data. Analyze this matrix using SVD to obtain the temporal and spatial eigen–modes and eigen values of the BPM response. Use the information from the eigen-vectors to determine some of the accelerator parameters at each BPM. Apply the knowledge of the transport between at least on pair of BPMs to determine the resonant excitation amplitude and all of the one turn transport parameters at these BPMs and to determine additional accelerator parameters at all of the BPMs. This analysis[1] is summarized below.

The data required for this analysis consists of two sets of BPM measurements taken for M BPMs on T sequential turns for each of the two dipole-betatron normal modes. Each data set is formed into a position-history matrix containing both x and y coordinates, having a size of T by 2M (for the M BPMs, sampled on T sequential turns.) The position-columns are arranged so that the order of elements begins with the x- and then y-position at the first BPM, and then through the last BPM. Each matrix element is divided by the square root of the number of temporal samples per BPM, T, giving the form,

$$\mathbf{P}^{T} = \frac{1}{\sqrt{T}}\left(\vec{p}_1, \ \vec{p}_2, \ \ldots, \ \vec{p}_T\right)$$

where the t-th row vector is

$$\vec{p}_t = \left(x_{1,t}, y_{1,t}, \ x_{2,t}, y_{2,t}, \ \ldots, \ x_{M,t}, y_{M,t}\right)$$

The SVD method[5] allows the rectangular matrix, P, to be written as a product of three matrices, T, Λ, and Π, (referred to, respectively, as the temporal, the eigen-value and the spatial matrices)

$$\mathbf{P} = \mathbf{T}\,\boldsymbol{\Lambda}\,\boldsymbol{\Pi}^{T} = \sum_{i}^{\text{modes}} \tau_i\,\lambda_i\,\pi_i$$

The rectangular matrix, Λ, of size T by 2M contains all zeroes except for a non-zero upper left-hand diagonal of singular values, λ_i. T and Π contain, respectively, ortho-normal temporal or spatial eigen-vectors for columns.

One turn motion in transversely coupled {4 x 4}, x-y coordinates may be decomposed into the two transverse normal coordinates, A-B, for the beam having the form,

$$\begin{pmatrix}\mathbf{M} & \mathbf{m}\\ \mathbf{n} & \mathbf{N}\end{pmatrix} = \mathbf{V}\mathbf{U}\mathbf{V}^{-1} = \begin{pmatrix}\gamma\mathbf{I} & -\mathbf{C}\\ \mathbf{C}^{+} & \gamma\mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{pmatrix}\begin{pmatrix}\gamma\mathbf{I} & \mathbf{C}\\ -\mathbf{C}^{+} & \gamma\mathbf{I}\end{pmatrix}$$

Using the normalized version of this form for coupling where $J_{A/B\,t}$ and $\phi_{A/Bt}$ are, respectively, the excitation actions and temporal phases on the t-th turn for the A/B normal modes, $\phi_{A/B\ m0}$ are phase advances for the A/B

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normal modes from a reference point 0 to detector m and $\overline{C}_{i\,j}$ is the i,j-th element of the normalized form of the C-matrix[4], the transpose of (x,y) position vector for the m-th BPM on the t-th turn (shown for both excitations) is

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix}_{t\,m} = \sqrt{\frac{2}{T}} \begin{pmatrix} \gamma_m \sqrt{\beta_{Am} J_{At}} \cos(\phi_{Am0} - \phi_{At}) \\ -\sqrt{\beta_{Am} J_{Bt}}\, \overline{C}_{11,m} \cos(\phi_{Bm0} - \phi_{Bt}) \\ +\sqrt{\beta_{Am} J_{Bt}}\, \overline{C}_{12,m} \sin(\phi_{Bm0} + \phi_{Bt}) \\ \gamma_m \sqrt{\beta_{Bm} J_{Bt}} \cos(\phi_{Bm0} + \phi_{Bt}) \\ +\sqrt{\beta_{Bm} J_{At}}\, \overline{C}_{22,m} \cos(\phi_{Am0} - \phi_{At}) \\ +\sqrt{\beta_{Bm} J_{At}}\, \overline{C}_{12,m} \sin(\phi_{Am0} - \phi_{At}) \end{pmatrix}$$

When either mode is excited, each of these components may be written in the form, $\Delta_m \sin(\phi_m + \delta_m - \phi_t)$.

To identify the four column eigen-vectors contained in Π, which describe the dipole motion of the beam in its two normal modes, construct the spatial covariance matrix, $\mathbf{K} = \mathbf{P}^T\ \mathbf{P}$, where the i-th and j-th element of K may be written as

$$\begin{aligned} K_{ij} &= \frac{2}{T}\sum_{t=1}^{T} J_t \Delta_i \Delta_j \sin(\phi_i + \delta_i - \phi_t)\sin(\phi_j + \delta_j - \phi_t) \\ &= \frac{1}{T}\sum_{t=1}^{T} J_t \Delta_i \Delta_j \left[\cos(\phi_i - \phi_j + \delta_i - \delta_j) - \cos(\phi_i + \delta_i + \phi_j + \delta_j - 2\phi_t)\right] \\ &= \langle J_t \rangle \Delta_i \Delta_j \cos(\phi_i - \phi_j + \delta_i - \delta_j) \end{aligned}$$

The last equality is correct, if the total number of turns, T, is sufficiently large that the second term of the second line averages to zero, giving a result with the action averaged over the T turns, $\langle J_t \rangle$. The K matrix has the property,

$$\mathbf{K\Pi} = \left(\mathbf{T\Lambda\Pi}^T\right)^T\left(\mathbf{T\Lambda\Pi}^T\right)\mathbf{\Pi} = \mathbf{\Pi\Lambda}^T\mathbf{\Lambda}$$

So using the k-th column vector in Π, $\vec{\pi}_k$, and since $\Lambda^T\Lambda$ is diagonal, this becomes an eigen-value equation,

$$\mathbf{K}\vec{\pi}_k = \lambda_k^2 \vec{\pi}_k$$

Solving the characteristic equation yields two solutions $\lambda_\pm$ and the following forms for the spatial eigen-vector,

$$\vec{\pi}_\pm^T = \left(\ldots,\ \frac{\sqrt{\langle J_t \rangle}}{\lambda_\pm} \Delta_i \begin{bmatrix} \sin \\ \cos \end{bmatrix} (\phi_i + \delta_i + \phi_{0,+}),\ \ldots \right)$$

where sin(cos) functions correspond to the +(-) subscript, and $\phi_{0,+}$ is a phase which corresponds to λ_+. Similarly the corresponding temporal eigen-vectors, $\tau_\pm$,

$$\vec{\tau}_\pm^{\,T} = \left(\ldots,\ \sqrt{\frac{2\ J_t}{\langle J_t \rangle T}} \begin{bmatrix} \text{-}\sin \\ \cos \end{bmatrix} (\phi_t + \phi_{0,+}),\ \ldots \right)$$

The i-th components of the spatial eigen-vectors give

$$\begin{aligned} \tan(\phi_i + \delta_i + \phi_{0,+}) &= \frac{\lambda_+\, \hat{i}^T \vec{\pi}_+}{\lambda_-\, \hat{i}^T \vec{\pi}_-} \\ \langle J_t \rangle \Delta_i^2 &= \left[\left(\lambda_+\, \hat{i}^T \vec{\pi}_+\right)^2 + \left(\lambda_-\, \hat{i}^T \vec{\pi}\right)^2 \right] \end{aligned}$$

where i^T represents the unit row vector with the i-th component equal to unity (to extract the i–th element of the eigen vector.) In these expressions $\phi_i + \delta_i$ contains the betatron phase advance and the tangent of the ratio of two of the $\overline{C}$ elements, while Δ_i has $\beta^{1/2}$ and the magnitude of the same two coupling-matrix elements and $\sqrt{\langle J_t \rangle}$ is set by the scale of the excitation amplitude. From these expressions ratios of β's at the BPMs are determined, however the inclusion of BPMs a known distance apart allows fixing the absolute scale of the β's[1].

ESTIMATED ACCURACY FROM SIMULATION STUDIES

The measurement errors of the BPMs will limit how accurately any of the accelerator parameters may be determined. One approach has been to calculate the minimum uncertainty in some particle accelerator parameters based on simply using the uncertainty for each BPM measurement[1]. These estimated uncertainties for the parameters are then assumed to require an additional scaling factor to yield the actual resulting uncertainties from an SVD analysis of data in the presence of the BPM errors, Δx. If the A dipole-mode is driven, the fractional error of beta-function for data sets of N measurements is

$$\frac{\delta(\gamma_m^2\ \beta_A)}{\gamma_m^2\ \beta_A} = 4\kappa \left(\frac{1}{\gamma_m\ \beta_A^{1/2}} \right) \frac{\Delta x}{\sqrt{N}\ J_{At}^{1/2}}$$

where κ is the scaling factor. Likewise, the error in the C-matrix elements may be established for the same dipole-mode using the displacements from the "out-of-plane" component of the oscillation (having BPM errors Δy) as

$$\delta\left(\beta_B^{1/2}\ \overline{C}_{12,m}^{22}\right) = 2\kappa' \frac{\Delta y}{\sqrt{N}\ J_A^{1/2}}$$

where κ' is the scaling factor and

$$\overline{C}_{12,m}^{22} \equiv \sqrt{\overline{C}_{22,m}^{\,2} + \overline{C}_{12,m}^{\,2}}$$

To place bounds on the values for κ and κ', simulations were undertaken using one set of CESR optics, CHESS_20050617, which contains a region of large coupling surrounding the CLEO solenoid magnet. The simulations yielded detailed comparisons of the design accelerator parameters vs. the parameters determined from the SVD analysis at 14 (of the 102 total) BPMs, and this is considered to be a representative sampling as the

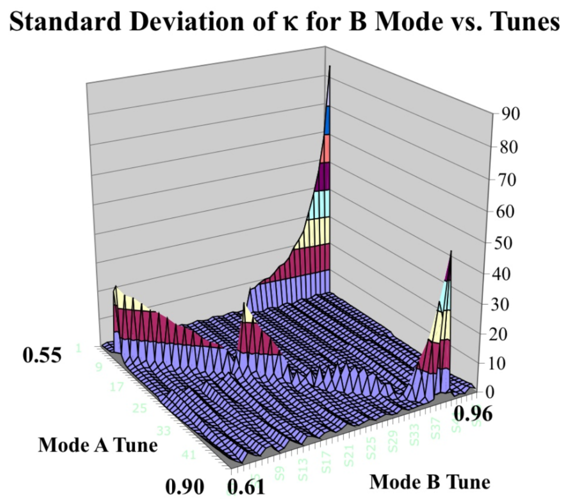

Figure 1. Plot of the standard deviation of a fit for κ at each tune plane point vs. tunes for mode B for 1024 turns of data.

set includes detectors within the coupling-compensation region around the high energy physics solenoid magnet.

To study the accuracy of the SVD method, we first examined the values of λ's that were calculated. The analysis algorithm typically produces five values larger than the others. The largest value for λ is associated with a spatial eigen vector, which is essentially a small constant offset for the position (even though the data set had its average value subtracted initially.) The next two larger values of λ are the λ_+ and λ_- values for the eigen mode of the beam that was intended to be excited. The second pair of eigen values are those associated with the beam's other eigen mode, which was excited slightly due to coupling. FFTs of the temporal eigen modes to find the oscillation frequency confirmed which of the eigen values of the four belong to which of these two pairs. The remaining eigen values cluster into two groups, each having different magnitudes. By changing the position noise added to the data, we determined that the set with the larger magnitudes scales with the noise amplitude, while the other set of eigen values have magnitudes representing the round-off errors in the calculations.

The analysis calculated the average value for κ and the standard deviation from the ratio of the calculated fractional error in $\gamma^2\beta$ and the known excitation and position noise levels for each BPM at each point in the tune plane for 1024 turns of data. Figure 1 is a plot of the standard deviation of the average value for κ for all BPM's at each point in the tune plane for the excitation of dipole mode B. Regions around the half integer and integer resonances have been removed from the plot, since generally large values for the standard deviation κ occur near the half integer, the integer or the coupling resonance; otherwise κ remains near unity with smaller undulations visible on a finer scale. The undulations occur because the second term, which was neglected, in the expression for K_{ij} (above) does not completely average to zero at all points within the tune plane, instead it scales as 1/T. To obtain a result that is typical for the vast majority of the area within the tune plane κ was averaged at all BPMs at all points in the tune plane away from the half integer, integer and coupling resonance by removing tune points when the value of κ deviated by more than 3 standard deviations from the average. By analyzing 1024 turns of data we obtained results for κ and κ' consistent with unity when both A- and B-modes were excited. Analyzing the same set of simulated data for 256 turns gives values for κ and κ' larger by approximately a factor of three, close to the expected scaling from the underlying interference pattern in the tune plane.

RESULTS OF MEASUREMENTS

Actual 1024 turn data was taken in CESR for a subset (8) of the BPMs, capable of turn-by-turn measurements, when the beam was excited in each of the betatron dipole modes of oscillation. At the same time an independent set of phase and coupling measurements were made as a reference for "actual" accelerator optics. From the known measurement uncertainties for these BPMs (30 μm) and the approximate excitation of 1 mm peak at a β's equaling 35 m, we estimate an expected uncertainty of 7×10^{-3} for $\overline{C}$ elements, Δβ/β and equivalently phase advance errors for these measurements. Using MIA, we compared differences in phase advances between BPMs and $\overline{C}$ elements between different sets of data and found RMS differences of 5.1×10^{-2} and 1.1×10^{-2} for the A- and B-mode excitations, respectively, and RMS differences of 1.4×10^{-2} for $\overline{C}_{12}$. Although the A-mode errors are larger than expected by a factor of 5, the other results are consistent with the expected uncertainties. However, when comparing the BPM-to-BPM phase advances between MIA data and the conventional phase measurements used at CESR, we found RMS differences of 1.5×10^{-1}-much larger than expected. Although we suspect this is an indication of possible systematic differences between these techniques, further study will wait until after the installation of turn-by-turn readouts for the remainder of the CESR BPMs is completed.

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