MEASURING BETATRON TUNES WITH DRIVEN OSCILLATIONS*

R. A. Bosch#, Synchrotron Radiation Center, University of Wisconsin-Madison, 3731 Schneider Dr., Stoughton, WI 53589, U.S.A.

Abstract

The betatron tunes of an electron storage ring may be measured by driving transverse oscillations with an excitation electrode and measuring the resonant beam response with a pickup electrode. We model the damping of coherent betatron oscillations from the tune spread and radiation damping, finding that the tune signal is proportional to the square root of the product of the betatron functions at the excitation and pickup locations. The signal is independent of the betatron phase advance between the two locations. Our results are applied to the Aladdin 800-MeV electron storage ring.

INTRODUCTION

To analyze the measurement of betatron tunes by using driven resonant transverse oscillations, the damping of coherent oscillations from the tune spread and radiation damping is considered. For a single-peak tune distribution that may be approximated by a Cauchy distribution, the phase-mix damping causes an exponential decay of coherent betatron oscillations, so that both the tune spread and radiation damping are approximated by exponential decay. For a small driven oscillation from a tune excitation electrode, we find an analytic formula for the beam response at the position of a tune pickup electrode.

DRIVEN OSCILLATIONS

Consider a driven horizontal or vertical oscillation in a storage ring with no horizontal-vertical coupling or chromaticity. A kick that bends the orbit through the angle θ at the longitudinal position $s=0$ at time $t=0$ gives excitations at the location $s\in[0,C)$, where C is the ring circumference, arriving at times $t=nT_0+s/c$, $n=0,1...\infty$, where T_0 is the recirculation time and c is the speed of light.

When the radiation damping and tune spread are neglected, the excitation arriving at time nT_0+s/c, given by eqs. (2.57) and (2.58) of Ref. [1], is

$$x_n(s)=\sqrt{\beta(0)}\sqrt{\beta(s)}\theta\sin[2\pi\nu_\beta n+\psi(s)]. \quad (1)$$

Here, $\beta(s)$ is the betatron function, ν_β is the betatron tune, and $\psi(s)$ is the betatron phase at the pickup location minus the betatron phase at the excitation location. With radiation-damping rate α_R, the excitation becomes

$$x_n(s)=\sqrt{\beta(0)}\sqrt{\beta(s)}\theta\sin[2\pi\nu_\beta n+\psi(s)]\exp[-\alpha_R(nT_0+s/c)]. \quad (2)$$

For tune spread resulting from a distribution of angular betatron frequencies $f(\omega_\beta)$, with nominal betatron frequency $\omega_{\beta 0}$, the complex decoherence function may be defined as [2] $\hat{D}(t)\equiv\int d\omega_\beta f(\omega_\beta)\exp[i(\omega_\beta-\omega_{\beta 0})t]$. The excitation of the beam centroid, including the effect of radiation damping and decoherence from tune spread is

$$x_n(s)=\sqrt{\beta(0)}\sqrt{\beta(s)}\theta\exp[-\alpha_R(nT_0+s/c)] \times\mathrm{Re}\left(\exp\{i[2\pi\nu_\beta n+\psi(s)-\pi/2]\}\hat{D}(nT_0+s/c)\right). \quad (3)$$

For time-dependent excitation $\theta(t)$, the response of the beam centroid is

$$x(s,t)=\sum_{n=0}^{\infty}\sqrt{\beta(0)}\sqrt{\beta(s)}\theta(t-nT_0-s/c) \times\exp[-\alpha_R(nT_0+s/c)] \times\mathrm{Re}\left(\exp\{i[2\pi\nu_\beta n+\psi(s)-\pi/2]\}\hat{D}(nT_0+s/c)\right). \quad (4)$$

For a Cauchy (also called Lorentzian) distribution of angular betatron frequencies with half-width $\delta\omega_\beta$, $f(\omega_\beta)=(\delta\omega_\beta/\pi)/[(\omega_\beta-\omega_{\beta 0})^2+(\delta\omega_\beta)^2]$, coherent oscillations decay exponentially from phase-mix damping [3] so that $\hat{D}(t)=\exp(-\delta\omega_\beta|t|)$. Thus, the beam response is

$$x(s,t)=\sum_{n=0}^{\infty}\{\sqrt{\beta(0)}\sqrt{\beta(s)}\theta(t-nT_0-s/c) \times\sin[2\pi\nu_\beta n+\psi(s)]\exp[-\alpha(nT_0+s/c)]\}. \quad (5)$$

where $\alpha=\alpha_R+\delta\omega_\beta$ is the total damping rate from radiation damping and tune spread.

For a periodic excitation $\theta(t)=\theta_0\cos(\omega t+\phi)$,

$$x(s,t)=\sum_{n=0}^{\infty}\{\sqrt{\beta(0)}\sqrt{\beta(s)}\theta_0\cos[\omega(t-nT_0-s/c)+\phi] \times\sin[2\pi\nu_\beta n+\psi(s)]\exp[-\alpha(nT_0+s/c)]\}. \quad (6)$$

Equation (6) may be written in complex notation as

$$x(s,t)=\theta_0\sqrt{\beta(0)}\sqrt{\beta(s)}e^{i(\omega t+\phi)} \times\sum_{n=0}^{\infty}e^{-i(\omega-i\alpha)(s/c+nT_0)}\sin[2\pi\nu_\beta n+\psi(s)], \quad (7)$$

where the real part of $x(s,t)$ is the physical oscillation.

Writing the sin term as the difference of two exponentials and using the relation $\omega_0 T_0=2\pi$, where ω_0 is the angular revolution frequency, we have

$$x(s,t)=(\theta_0/2i)\sqrt{\beta(0)}\sqrt{\beta(s)}e^{i(\omega t+\phi)}e^{-i(\omega-i\alpha)s/c} \times\left(e^{i\psi(s)}\sum_{n=0}^{\infty}e^{-i(\omega-i\alpha-\nu_\beta\omega_0)nT_0}-e^{-i\psi(s)}\sum_{n=0}^{\infty}e^{-i(\omega-i\alpha+\nu_\beta\omega_0)nT_0}\right). \quad (8)$$

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bosch@src.wisc.edu

Using $\sum_{n=0}^{\infty} e^{\kappa n} = (1-e^{\kappa})^{-1}$ gives

$$x(s,t) = (\theta_0/2i)\sqrt{\beta(0)}\sqrt{\beta(s)}e^{i(\omega t+\phi)}e^{-i(\omega-i\alpha)s/c} \times\left[\frac{e^{i\psi(s)}}{1-e^{-i(\omega-\nu_\beta\omega_0-i\alpha)T_0}} - \frac{e^{-i\psi(s)}}{1-e^{-i(\omega+\nu_\beta\omega_0-i\alpha)T_0}}\right], \quad (9)$$

which may be written as

$$x(s,t) = (-\theta_0/4)\sqrt{\beta(0)}\sqrt{\beta(s)}e^{i(\omega t+\phi)}e^{-i(\omega-i\alpha)s/c}e^{\alpha T_0/2} \times\left[\frac{e^{i\psi(s)+i(\omega-\nu_\beta\omega_0)T_0/2}}{\sin[(\omega-\nu_\beta\omega_0-i\alpha)T_0/2]} - \frac{e^{-i\psi(s)+i(\omega+\nu_\beta\omega_0)T_0/2}}{\sin[(\omega+\nu_\beta\omega_0-i\alpha)T_0/2]}\right]. \quad (10)$$

Defining the tune of the driving frequency ω as $\nu_d = \omega T_0/2\pi$ gives

$$x(s,t) = (-\theta_0/4)\sqrt{\beta(0)}\sqrt{\beta(s)}e^{i(\omega t+\phi)}e^{-i(\omega-i\alpha)s/c}e^{\alpha T_0/2} \times\left[\frac{e^{i\psi(s)+i\pi(\nu_d-\nu_\beta)}}{\sin[\pi(\nu_d-\nu_\beta)-i\alpha T_0/2]} - \frac{e^{-i\psi(s)+i\pi(\nu_d+\nu_\beta)}}{\sin[\pi(\nu_d+\nu_\beta)-i\alpha T_0/2]}\right]. \quad (11)$$

Equation (11) is our main result. The real part of Eq. (11) describes driven oscillations with the effects of radiation damping and the tune spread of a Cauchy distribution.

When the values of $\nu_d \pm \nu_\beta$ differ from all integers by much more than $\alpha T_0/2\pi$ (which requires that $\alpha T_0/2\pi << 1$), the damping may be neglected. In this case, eq. (11) becomes

$$x(s,t) = (-\theta_0/4)\sqrt{\beta(0)}\sqrt{\beta(s)}e^{i(\omega t+\phi)}e^{-i\omega s/c} \times\left[\frac{e^{i\psi(s)+i\pi(\nu_d-\nu_\beta)}}{\sin[\pi(\nu_d-\nu_\beta)]} - \frac{e^{-i\psi(s)+i\pi(\nu_d+\nu_\beta)}}{\sin[\pi(\nu_d+\nu_\beta)]}\right], \quad (12)$$

in agreement with the ac-dipole theory of Refs. [4–7], after accounting for missing phase factors. When damping can be neglected and $\omega = n\,\omega_0$ (so that $\nu_d = n$), we have

$$x(s,t) = (\theta_0/2)\sqrt{\beta(0)}\sqrt{\beta(s)}e^{i(\omega t+\phi)}e^{-i\omega s/c} \times\cos[\psi(s)-\pi\nu_\beta]/\sin(\pi\nu_\beta). \quad (13)$$

For the special case $\omega = \phi = 0$ (so that $\nu_d = 0$), eq. (13) reproduces the closed-orbit disturbance

$$x(s,t) = (\theta_0/2)\sqrt{\beta(0)}\sqrt{\beta(s)}\cos[\psi(s)-\pi\nu_\beta]/\sin(\pi\nu_\beta), \quad (14)$$

given by eq. (2.92) of Ref. [1], [where eq. (2.92) is corrected, according to the SLAC-121 Addendum of May 1979, by multiplying by -1]. In eqs. (13) and (14), the amplitude depends upon the betatron phase advance $\psi(s)$ between the excitation location and the pickup location.

Now consider the measurement of betatron tunes by resonant excitation, where the damping from radiation damping and tune spread cannot be neglected. When $\nu_d - \nu_\beta \approx n$ for some integer n, while $\nu_d + \nu_\beta$ differs from all integers by much more than $|\nu_d - \nu_\beta - n|$, the first term in eq. (11) dominates so that

$$x(s,t) \approx (-\theta_0/4)\sqrt{\beta(0)}\sqrt{\beta(s)}e^{i(\omega t+\phi)}e^{-i\omega s/c} \times\left[\frac{e^{i\psi(s)+i\pi(\nu_d-\nu_\beta)}}{\sin[\pi(\nu_d-\nu_\beta)-i\alpha T_0/2]}\right]. \quad (15)$$

The amplitude of the oscillation at driving tune ν_d is

$$(\theta_0/4)\sqrt{\beta(0)}\sqrt{\beta(s)}\,/\,|\sin[\pi(\nu_d-\nu_\beta)-i\alpha T_0/2]|\,. \quad (16)$$

The largest oscillation occurs for resonant excitation with $\nu_d - \nu_\beta = n$, with amplitude

$$\theta_0\sqrt{\beta(0)}\sqrt{\beta(s)}\,/\,2\alpha T_0\,. \quad (17)$$

For resonant excitation, the amplitude at the pickup does not depend upon the betatron phase advance $\psi(s)$.

The above analysis has considered a periodic single-frequency oscillation. When the tune is measured by sweeping the excitation angular frequency at rate $d\omega/dt$, a rule of thumb for a swept-frequency spectrum analyzer indicates that the measurement bandwidth $\Delta\omega_m$ obeys $(\Delta\omega_m)^2 \approx 2\pi\, d\omega/dt$. For spectral features with width $\Delta\omega >> \Delta\omega_m \approx \sqrt{2\pi\, d\omega/dt}$, the response obeys the above theory for a periodic single-frequency oscillation.

The minimum sweep time for measurement of a feature with bandwidth $\Delta\omega$ occurs when the frequency is swept over a range $\sim\Delta\omega$ in time τ with $(\Delta\omega)^2 \approx 2\pi\Delta\omega/\tau$, i.e. $\tau \sim 2\pi/\Delta\omega$. Therefore, measuring the tune peak and its width at $\nu_d = \nu_\beta$, where $\Delta\omega \approx \alpha$, requires a minimum sweep time τ of $\sim 2\pi/\alpha$.

Measuring the beam response at a driving tune $\nu_d \approx \nu_\beta$, where $\Delta\omega \approx 2\,|\sin[\pi(\nu_d-\nu_\beta)-i\alpha T_0/2]|\,/\,T_0$, requires sweep time $\sim 2\pi T_0/(2\,|\sin[\pi(\nu_d-\nu_\beta)-i\alpha T_0/2]|)$. This is comparable to the required measurement time when ramping the amplitude of a single-frequency excitation [6].

ALADDIN

The betatron tunes of the Aladdin 800-MeV electron storage ring are measured with excitation and pickup electrodes oriented at 45° with respect to the horizontal direction. To install an undulator, the excitation electrode was moved to a new location where the values of $\sqrt{\beta_x}$ and $\sqrt{\beta_y}$ are within 30% of their values at the old location. Moving the tune excitation electrode is predicted to modify the tune signals by less than 30%, which is inconsequential for ordinary operations. As expected, moving the excitation electrode did not impact the measurement of betatron tunes.

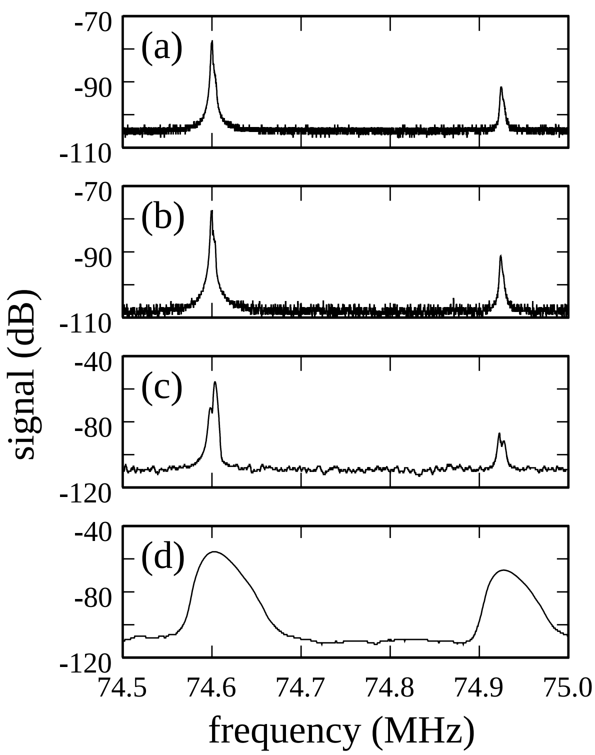

Figure 1. Aladdin tune measurements performed by sweeping the frequency of driven transverse oscillations at a constant rate. (a) Sweep time = 10 s. (b) Sweep time = 1 s. (c) Sweep time = 100 ms. (d) Sweep time = 10 ms.

The radiation damping time constants are ~30 ms, while the decoherence of betatron oscillations occurs over ~1000 turns = 0.3 ms [8]. The damping rate from both radiation damping and tune spread is $\alpha \sim (0.3\,\text{ms})^{-1}$. Thus, the sweep time required for measurement of a single tune peak and its width is $\sim 2\pi \times 0.3$ ms.

For the horizontal tune peak with $\alpha \sim (0.3\,\text{ms})^{-1}$, $T_0 = 2.96\times10^{-7}$ s and $\beta_x(0) \sim \beta_x(s) \sim 4$ m, eq. (17) gives a peak oscillation amplitude (in meters) of $2000\,\theta_0$, where θ_0 is the peak deflection from the excitation electrode in radians. For vertical tune measurement with $\beta_x(0) \sim \beta_x(s) \sim 0.7$ m, the peak oscillation amplitude is $350\,\theta_0$. For a given value of θ_0, the horizontal peak is expected to exceed the vertical peak by $20\log(2000/350) = 15$ dB.

To measure both tune peaks and their widths with a constant-rate sweep over the angular frequency range $\omega_0(\nu_y - \nu_x) = (2\pi)(3.4\,\text{MHz})(7.234-7.139) = 2\times10^6$ rad/s, we expect that the minimum required sweep time τ obeys $d\omega/dt = (2\times10^6\ \text{rad/s})/\tau \approx (\Delta\omega)^2/2\pi \approx (0.3\,\text{ms})^{-2}/2\pi$, i.e. $\tau \approx 1$ s.

We operated the Aladdin storage ring with approximately zero chromaticity by tuning the two sextupole families near the thresholds of horizontal and vertical head-tail instability. With zero chromaticity, the horizontal and vertical tune signals are single peaks that can be approximated by our analysis of Cauchy distributions of betatron frequencies. Measurements shown in Fig. 1 confirm that a minimum sweep time of ~1 s is required to measure both tune peaks and their widths, while the horizontal peak exceeds the vertical peak by 15 dB.

SUMMARY

We analyzed the measurement of betatron tunes by driving transverse oscillations with an excitation electrode and detecting the beam response with a pickup electrode. We considered radiation damping and the phase-mix damping of Cauchy distributions of betatron frequencies.

For resonant excitation, the tune signal is proportional to the square root of the product of the betatron functions at the excitation and pickup locations. The signal is independent of the betatron phase advance between the two locations.

For measurement of the tune peaks and their widths at the Aladdin 800-MeV electron storage ring, our analysis successfully predicts the minimum required sweep time and the ratio of the horizontal and vertical peak heights.

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