NOTE ON SOME THERMAL ANALYTIC SOLUTIONS IN ACCELERATOR ENGINEERING

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Abstract

A bending magnet, a linear undulator, an elliptically polarized undulator and a wiggler all generate synchrotron radiation that accelerator engineers encounter while they are designing components with a great heat load. Because their power distributions belong to a characteristic type, some analytic solutions for temperature are available and can serve for a parametric study. Furthermore, these closed-form solutions provide an optimized tool applicable to thermo-mechanical design such as for the crotch absorber, fixed masks, photon absorbers, mirrors or other subsystems with large heat loads. These solutions also indicate a simplified implementation applicable to finite-element modelling in real cases.

INTRODUCTION

Taiwan Photon Source (TPS) is a new synchrotron accelerator project of National Synchrotron Radiation Research Center (NSRRC) in Taiwan. A 3-GeV, 400-mA high-energy third-generation synchrotron accelerator will be constructed and is planned to be commissioned in 2013. Because of the distribution of its great thermal power in both a bending magnet and the insertion devices, an extensive thermo-mechanical analysis is performed before design. Currently, the finite-element method (FEM) is commonly used to calculate temperatures and thermally induced stress, but the distribution functions for a bending magnet and an insertion device are typically of Gaussian type that is difficult to implement with the FEM 2D/3D model. In general, an engineer meshes the areas on which the power is incident into a small grid (normally at least 10 grids within the FWHM heating region), then inputs the corresponding Gaussian power magnitude on the gridded nodes. Two drawbacks are known.

- In most cases, the heated spot is much smaller than that of the entire solid model. To model a Gaussian profile added to the heated area, very fine meshes must be generated along these areas. This condition produces complication in generating the mesh and results in a large aspect ratio of the element.
- How fine an element is sufficient? This question always arises on meshing the model on the heated surface on which the synchrotron radiation is incident. Even when an engineer generates a fine mesh along the entire heated surface, he/she must still verify that the total power deposited in the model is consistent with the theoretical value. The testing procedure is to sum all power density deposited on the entire elemental heating surface for comparison with the theoretical total power input. In general, with most commercial FEM packages this task is not straightforward.

Furthermore, to ensure the qualitative correctness of the FEM result, one should verify the result by other means, such as an analytic solution or even a simple manual calculation. An analytic solution with a simplified model is an effective tool not only to verify the numerical results, but also to render an overall view of the entire problem; with variation of the parameters, one can perform parametric tests that yield an optimized design.

In this paper, we provide analytic solutions for the heating problem or power distribution of a typical bending magnet, undulator, wiggler and elliptical undulator. Some simplifications are made to preserve the representation but not to complicate unduly the problems. An analytic solution for the constant power distribution is also presented which indicates that one need not implement true Gaussian heating in modelling the problem in FEM.

BENDING MAGNET

The shape function of the power distribution of a bending magnet (BM) is given as [1]

$$q\left[\frac{Kw}{mrad^2}\right]=5.425E[GeV]B[T]I[mA]f(\gamma\varphi) \quad (1)$$

where

$$f(\gamma\varphi)=\frac{1}{\left(1+\gamma^2\varphi^2\right)^{5/2}}\left(1+\frac{5}{7}\frac{\gamma^2\varphi^2}{\left(1+\gamma^2\varphi^2\right)}\right) \quad (2)$$

relativistic energy $\gamma=1957E$, and φ is the vertical opening angle. Assuming a shape function, $f(\gamma\varphi)$ can be replaced by a distribution of Gaussian type:

$$f(\gamma\varphi)=\exp\left(-\frac{\gamma^2\varphi^2}{2\sigma_0^2}\right) \quad (3)$$

As the total linear power density should be in equilibrium, the value of standard deviation σ_0 must satisfy

$$\int_{-\infty}^{\infty}\frac{1}{\left(1+\gamma^2\varphi^2\right)^{5/2}}\left(1+\frac{5}{7}\frac{\gamma^2\varphi^2}{\left(1+\gamma^2\varphi^2\right)}\right)d(\gamma\varphi)= \int_{-\infty}^{\infty}\exp\left(-\frac{\gamma^2\varphi^2}{2\sigma_0^2}\right)d(\gamma\varphi) \quad (4)$$

The result yields

$$\frac{32}{21} = \sqrt{2\pi}\sigma_0 \Rightarrow \sigma_0 \approx 0.608 \tag{5}$$

which is identical to what Kim suggested [2]. The power distribution of a bending magnet is of Gaussian type along the vertical axis; the fan sweeps over 7.5^o horizontally per dipole. Decreasing the incident angle is one design concept to decrease the power density; it is therefore common to tilt the heating surface at a small horizontal angle. The size of the heated spot along the longitudinal direction (beam direction) is thus much larger than that in the vertical direction; any slice of the temperature distribution along a transverse cross section within this region is somewhat similar. Therefore a 2D model is sufficient for the analysis. A convective boundary condition is assumed on the other side to represent water cooling, and an adiabatic condition is assumed for the other surfaces.

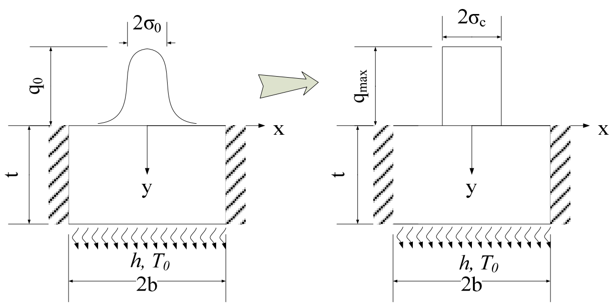

Figure 1: Gaussian heating problem maps to a piecewise constant heating problem.

The 2D steady-state heat equation states that

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{6}$$

with boundary conditions

$$\begin{aligned} &-k\frac{\partial T}{\partial y}(x,0) = q(x), \quad -k\frac{\partial T}{\partial x}(0,y) = 0, \\ &-k\frac{\partial T}{\partial y}(x,t) = h(T - T_\infty), \quad -k\frac{\partial T}{\partial x}(b,y) = 0 \end{aligned} \tag{7}$$

The temperature solution is given as [3]

$$\begin{aligned} \theta(\xi,\eta) = C_o\left[\eta - 1 - \frac{1}{Bi}\right] + \sum_{m=0}^{\infty} C_m \exp(-\lambda_m \eta)\cos(\lambda_m \xi) \\ \left[1 + \frac{\lambda_m - Bi}{\lambda_m + Bi}\exp(2\lambda_m(\eta - 1))\right] \end{aligned} \tag{8}$$

in which

$$\theta = \frac{k(T - T_0)}{q_o t}, \quad \xi = \frac{x}{t}, \quad \eta = \frac{y}{t}, \quad \beta = \frac{b}{t}, \quad Bi = \frac{ht}{k} \tag{9}$$

and $\lambda_m = \frac{m\pi}{\beta}$.

$$\begin{aligned} C_o &= -\frac{1}{\beta}\int_0^\beta q(\xi)d\xi, \\ C_m &= \frac{2\int_0^\beta q(\xi)\cos(\lambda_m \xi)d\xi}{\lambda_m \beta\left(1 - \exp(-2\lambda_m)\frac{\lambda_m - Bi}{\lambda_m + Bi}\right)} \end{aligned} \tag{10}$$

Applying a Gaussian power distribution produces

$$q(x) = q_o \exp\left(-\frac{x^2}{2\sigma_o^2}\right) \tag{11}$$

Coefficients C_0 and C_m in equation(10) are given in [3]. The corresponding maximum temperature is at the origin (0, 0). As the size of the vertical opening is much smaller than the heating block, an approximation using a piecewise constant heating profile is used, i.e.,

$$q(x) = q_{\max}(H(x - \sigma_c) - H(x + \sigma_c)) \tag{12}$$

in which appears *Heaviside function* $H(x)$; $q_{\max}$ and σ_c are the corresponding maximum power density and *'effective half-beam size'*, respectively. As the total power deposited on the heating boundary remains in equilibrium, equations (11) and (12) dictate that

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma_0^2}\right)dx = q_o\sqrt{2\pi}\sigma_o = \\ &\int_{-\infty}^{\infty} q_{\max}(H(x - \sigma_c) - H(x + \sigma_c))dx = 2q_{\max}\sigma_c \end{aligned} \tag{13}$$

Two possible options can be made; they are either

$$\text{(B) } Let \quad q_{\max} = q_o \Rightarrow \sigma_c = \sqrt{\frac{\pi}{2}}\sigma_o = \sqrt{\frac{\pi}{2}}\frac{0.608}{\gamma} \approx \frac{0.762}{\gamma} \tag{14}$$

or

$$\text{(C) } Let \quad \sigma_c = \sigma_o = \frac{0.608}{\gamma} \Rightarrow q_{\max} = \sqrt{\frac{\pi}{2}}q_o \approx 1.25q_o \tag{15}$$

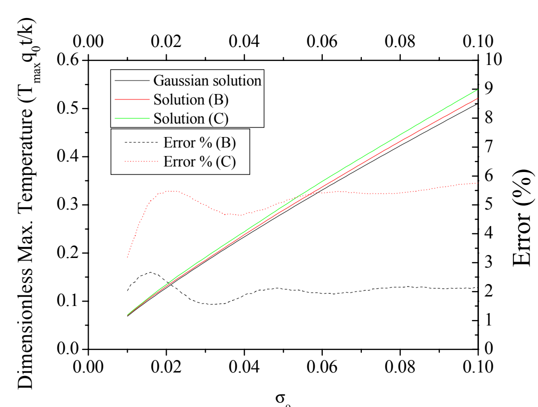

Figure 2: Gaussian power solution vs. constant power solutions.

Figure 2 shows the dimensionless maximum temperature and the discrepancies between solutions (B), (C) and the Gaussian type, Solution (B) is found to have only 2 % error whereas (C) is about 5 %.

UNDULATOR

In a linear undulator, if deflection parameters k for both x and y directions are almost identical, the power distribution can be approximated as an asymmetrical function. If the cooling block is also a circular one and relatively thin, the heating problem becomes simplified to a one-dimensional asymmetrical problem

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}-\frac{h}{kt}T+\frac{q_0}{kt}f(r)=0 \qquad (16)$$

in which $f(r)$ represents the shape of the power. Assuming no heat flow on the boundary $(r=r_0)$ and that it remains finite when $r=0.$, the solution of equation (16) is given as

$$\frac{h}{q_0}T=K_0(R)\int_0^R I_0(\tau)\tau f(\tau)d\tau+I_0(R)\frac{K_1(R_0)}{I_1(R_0)}$$
$$\int_0^{R_0} I_0(\tau)\tau f(\tau)d\tau+I_0(R)\int_R^{R_0} K_0(\tau)\tau f(\tau)d\tau \qquad (17)$$

in which $R_0=r_0\sqrt{\frac{h}{kt}}$, $R=r\sqrt{\frac{h}{kt}}$. $I_\nu(x)$ and $K_\nu(x)$ are *Modified Bessel functions or order* ν. An asymmetrical Gaussian power distribution simulated for the undulator can be expressed as

$$f(\tau)=\exp\left(\frac{kt\tau^2}{2h\sigma_0^2}\right) \qquad (18)$$

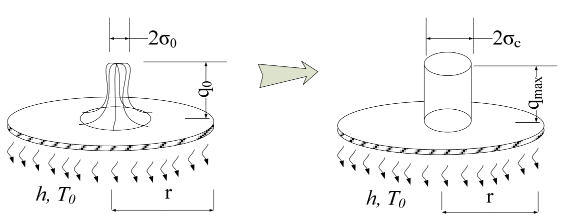

Figure 3: Asymmetrical Gaussian heating problem maps to piece-wise constant heating problem.

Similarly, if a piecewise constant heating power replaces a Gaussian type as illustrated in Figure 3, we must have

$$\int_0^\infty 2\pi q_0\exp\left(-\frac{r^2}{2\sigma_0^2}\right)rdr=2\pi q_0\sigma_0^2=$$
$$\int_0^\infty q_{\max}\pi\left[H(r)-H(r-\sigma_c)\right]rdr=q_{\max}\pi\sigma_c^2 \qquad (19)$$

Again, two possible options are either

$$\text{(B) } Let\ q_{\max}=q_o\Rightarrow\sigma_c=\sqrt{2}\sigma_o=\sqrt{2}\frac{0.608}{\gamma}\approx\frac{0.86}{\gamma} \qquad (20)$$

or

$$\text{(C) } Let\quad \sigma_c=\sigma_o=\frac{0.608}{\gamma}\Rightarrow q_{\max}=2q_o \qquad (21)$$

Figure 4 shows dimensionless maximum temperature rise and fractional error between piecewise constant heating and a Gaussian heating profile.

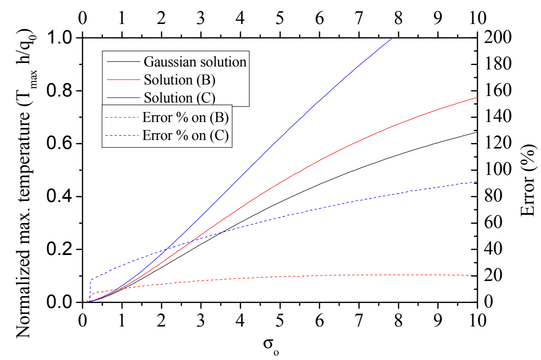

Figure 4: Asymmetrical Gaussian heating solution vs. piecewise heating solution.

For case (B) the overall average relative error is about 20 % whereas (C) is 60 %, and, as the beam size increases, their discrepancies increase correspondingly.

DISCUSSION AND CONCLUSION

Several analytic temperature solutions related to power distributions for a bending magnet and an undulator have been solved and are presented. The analysis reveals that a simplified, piecewise constant, solution for the power temperature gives a satisfactory approximation in comparison with that of Gaussian type. While performing FEM modelling, instead of implementing a fine mesh and the associated complicated nodal-surface Gaussian heating values, one need only model the heating surface with much less meshed element and, subsequently, implement only a constant value heat flux $q_{\max}$ within σ_c areas.

REFERENCES

[1] J. D. Jackson, "Classical Electrodynamics", Wiley, New York USA, second edition, 1975.

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[3] I. C. Sheng and T. Nian, "Some Closed-Form Solutions of the Temperature Field due to Bending Magnet and Undulator Heating in APS", High Heat Flux Engineering II, Vol. 1997 SPIE, Page 338-450.