# HEAD-TAIL MODES FOR STRONG SPACE CHARGE

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*Abstract*

The head-tail modes are described for the space charge tune shift significantly exceeding the synchrotron tune. A general equation for the modes is derived. The spatial shapes of the modes, their frequencies, and coherent growth rates are explored. The Landau damping rates are also found.

## INTRODUCTION

The head-tail instability of bunched beams was observed and theoretically described many years ago [1-3]. Since then, this explanation has been accepted and included in textbooks [4,5], but still there is an important gap in the theory of head-tail interaction. This relates to the influence of space charge on the coherent modes: their shapes, growth rates and Landau damping. In Ref. [6], an analytical description of the coherent modes was found for a square well model, air-bag distribution and a short-range wake function. Later, the air-bag limitation was removed for zero-wake case in the square well [7]. Compared to Ref. [6,7], an attempt of this paper is both broader and narrower. It is broader since there are no assumptions about the shape of the potential well, the bunch distribution function, and the wake function. The Landau damping is calculated in this paper. From another aspect, my approach is narrower than that of Ref. [6], since it is assumed that the space charge tune shift in the bunch 3D center $Q_{\max}$ is large compared to both to the synchrotron tune $Q_s$ and the wake-driven coherent tune shift $Q_w$: $Q_{\max} >> Q_s, Q_w$.

More detailed version of this paper is given by Ref. [10].

## RIGID SLICE EQUATIONS

Let $\theta$ be the time in radians, $\tau$ - a distance along the bunch in radians as well, $X_i(\theta)$ - a betatron offset of $i$-th particle, and $\bar{X}(\theta,\tau)$ - an offset of the beam center at the given time $\theta$ and position $\tau$. The bare betatron tune $Q_b$ can be excluded by using slow variables $x_i(\theta)$:

$$X_i(\theta) = \exp(-iQ_b\theta)x_i(\theta).$$

A single-particle equation of motion can be written as

$$\dot{x}_i(\theta) = iQ(\tau_i(\theta))\left[x_i(\theta) - \bar{x}(\theta,\tau_i(\theta))\right] - i\zeta v_i(\theta)x_i(\theta) - i\kappa\hat{\mathbf{W}}\bar{x}. \quad (1)$$

Here $\dot{x}_i = dx_i/d\theta$; $\zeta = -\xi/\eta$ with $\xi = dQ_b/d(\Delta p/p)$, $\eta = \gamma_t^{-2} - \gamma^{-2}$, $Q(\tau)$ is the space charge tune shift as a function of the position inside the bunch $\tau$, $v_i(\theta) = \dot{\tau}_i(\theta)$, and $\kappa\hat{\mathbf{W}}\bar{x}$ is the wake force expressed in terms of the wake linear operator $\hat{\mathbf{W}}$ to be specified below.

Eq. (1) assumes a rigid-slice approximation. This approximation is based on the idea that the transverse coherent motion of the beam can be treated as displacements of beam longitudinal slices, so the force on a given particle is just proportional to its offset from the local beam centroid. For a coasting beam, the validity of the rigid-slice model is discussed in Ref. [8]. The rigid-slice model requires a sufficient separation between the coherent frequency and the incoherent spectrum: the separation has to be significantly larger than the width of the bare incoherent spectrum. As a result almost all the particles respond almost identically to the collective field.

The chromaticity term can be excluded from Eq. (1) with a substitution $x_i(\theta) = y_i(\theta)\exp(-i\zeta\tau_i(\theta))$, leading to

$$\dot{y}_i(\theta) = iQ(\tau_i(\theta))\left[y_i(\theta) - \bar{y}(\theta,\tau_i(\theta))\right] - i\kappa(\hat{\mathbf{W}}\bar{y} + \hat{\mathbf{D}}\bar{y}) \quad (2)$$

with

$$\kappa = \frac{r_0 R}{4\pi\beta^2\gamma Q_b};$$

$$\hat{\mathbf{W}}\bar{y} = \int_\tau^\infty W(\tau - s)\exp(i\zeta(\tau - s))\rho(s)\bar{y}(s)ds;$$

$$\hat{\mathbf{D}}\bar{y} \equiv \bar{y}(\tau)\int_\tau^\infty D(\tau - s)\rho(s)ds. \quad (3)$$

Here $r_0$ is the classical radius of the beam particles; $R=C/(2\pi)$ is the average ring radius; $\beta$ and $\gamma$ are the relativistic factors, $\rho(s)$ is the bunch linear density normalized on the number of particles in the bunch, $\int \rho(s)ds = N_b$. The wake-function $W(s)$ is defined according to Ref [4] (slightly different from the definition of Ref. [6]), detuning function $D(s)$ is defined according to Ref. [9].

We begin from solving Eq. (2) for the no-wake case. Next the wake is taken into account as a perturbation of the space charge eigen-modes. These unperturbed eigen-modes are to be found from the no-wake reduction of Eq. (2):

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$$\dot{y}_i(\theta) = iQ(\tau_i(\theta))[y_i(\theta) - \bar{y}(\theta, \tau_i(\theta))] \quad (4)$$

Solutions of this equation give the space charge eigen-modes: their spatial shapes and frequencies. The modes do not depend on the chromaticities, except for the common head-tail phase factor $\exp(-i\zeta\tau)$. The chromaticity enters into the problem through the wake term, Eq. (3), affecting the coherent growth rates. As it will be seen below, the chromaticity normally makes the coherent growth rates negative for the modes, which number $k$ is smaller than the head-tail phase, $k \lesssim \zeta\sigma$, with $\sigma$ as the rms bunch length.

## GENERAL SPACE CHARGE MODES

In this section, an ordinary differential equation for the eigen-modes is derived for a general potential well and 3D bunch distribution function, assuming strong space charge, $Q >> kQ_s$. As a result, all the individual degrees of freedom are detuned from the coherent motion by approximately the same number, namely, the local space charge tune shift. Consequently, locally all the particles are moving almost identically. That is why the space charge modes are described by single-argument functions dependent on the position along the bunch only.

The single-particle equation (4) can be solved in general:

$$y_i(\theta) = -i\int_{-\infty}^{\theta} Q(\tau_i(\theta'))\bar{y}(\theta', \tau_i(\theta'))\exp(i\Psi(\theta) - i\Psi(\theta'))d\theta'; \quad (5)$$
$$\Psi(\theta) \equiv \int_0^{\theta} Q(\tau_i(\theta'))d\theta' .$$

Since $Q(\tau) > 0$, $\Psi(\theta)$ is monotonic and so integration over $\theta$ in Eq. (5) can be replaced by integration over $\Psi$:

$$y_i(\Psi) = -i\int_{-\infty}^{\Psi} \bar{y}(\Psi')\exp(i\Psi - i\Psi')d\Psi' . \quad (6)$$

When the space charge tune shift is so high that $Q >> kQ_s$, the phase $\Psi$ runs fast compared with relatively slow dependence $\bar{y}(\Psi)$, so the later can be expanded in a Taylor series:

$$\bar{y}(\Psi') \approx \bar{y}(\Psi) - (\Psi - \Psi')\frac{d\bar{y}}{d\Psi} + \frac{(\Psi - \Psi')^2}{2}\frac{d^2\bar{y}}{d\Psi^2} .$$

After that the integral is easily evaluated:

$$y_i(\Psi) = \bar{y}(\Psi) - i\frac{d\bar{y}}{d\Psi} - \frac{d^2\bar{y}}{d\Psi^2} . \quad (7)$$

To come back to original variables, one can use that

$$\frac{d}{d\Psi} = \frac{\mathrm{v}}{Q(\tau)}\frac{\partial}{\partial\tau} + \frac{1}{Q(\tau)}\frac{\partial}{\partial\theta} = \frac{1}{Q(\tau)}\left(\mathrm{v}\frac{\partial}{\partial\tau} - i\nu\right). \quad (8)$$

Applied to Eq. (7), this gives

$$y_i = \bar{y} - \frac{i}{Q(\tau)}\left(\mathrm{v}\frac{\partial}{\partial\tau} - i\nu\right)\bar{y} - \left[\frac{1}{Q(\tau)}\left(\mathrm{v}\frac{\partial}{\partial\tau} - i\nu\right)\right]^2\bar{y} . \quad (9)$$

Now we can average over velocities v at the given position $\tau$, neglecting the eigenvalue $\nu_k$ in the second-order term of Eq. (9). After that, the equation for eigen-modes follows as a second-order ordinary self-adjoint differential equation:

$$\nu\bar{y} + u(\tau)\frac{d}{d\tau}\left(\frac{u(\tau)}{Q(\tau)}\frac{d\bar{y}}{d\tau}\right) = 0;$$
$$u^2(\tau) \equiv \frac{\int_{-\infty}^{\infty}\mathrm{v}^2 f(\mathrm{v},\tau)d\mathrm{v}}{\int_{-\infty}^{\infty} f(\mathrm{v},\tau)d\mathrm{v}}, \quad (10)$$

where $f(\mathrm{v},\tau)$ is a normalized steady-state longitudinal distribution function, $f(\mathrm{v},\tau) = f(H(\mathrm{v},\tau))$, with $H(\mathrm{v},\tau)$ as the longitudinal Hamiltonian.

Actually, Eq. (10) is valid for any beam transverse distribution, after certain redefinition of the space charge tune shift $Q(\tau)$. Indeed, the single-particle Eq. (9) does not make any assumption about the individual space charge tune shift dependence $Q(\tau)$, which can be considered as dependent on the transverse actions $J_{1i}, J_{2i}$ as well: $Q(\tau) \to Q_i(\tau) = Q(J_{1i}, J_{2i}, \tau)$. The averaging of Eq. (9) just has to take into account this dependence of the space charge tune shift on the transverse actions. As an example, for a Gaussian round beam, i.e. a beam with identical emittances and beta-functions, the transverse dependence of the space charge tune shift can be calculated as [8]:

$$Q(J_1, J_2, \tau) = Q_{\max}(\tau)\int_0^1 \frac{\left[\mathrm{I}_0\left(\frac{J_1 z}{2}\right) - \mathrm{I}_1\left(\frac{J_1 z}{2}\right)\right]\mathrm{I}_0\left(\frac{J_2 z}{2}\right)}{\exp(z(J_1 + J_2)/2)}dz \equiv \quad (11)$$
$$\equiv Q_{\max}(\tau)g(J_1, J_2)$$

Here $J_1, J_2$ are two dimensionless transverse actions, conventionally related to the offsets as $x = \sqrt{2J_1\varepsilon_1\beta_1}\cos(\psi)$ with $\varepsilon_1$ and $\beta_1$ as the rms emittance and beta-function, so that the transverse distribution function is $f_\perp(J_1, J_2) = \exp(-J_1 - J_2)$.

The transverse averaging of Eq. (9) requires calculation of two transverse moments $q_{-1}, q_{-2}$ of the tune shift $Q(J_1, J_2, \tau)$ generally defined by:

$$\left\langle Q_i^p(\tau)\right\rangle_\perp = \int_0^\infty\int_0^\infty dJ_1 dJ_2 f_\perp(J_1, J_2)Q^p(J_1, J_2, \tau) \equiv q_p^p Q_{\max}^p(\tau) .$$
$$q_p = \left[\int_0^\infty\int_0^\infty dJ_1 dJ_2 f_\perp(J_1, J_2)g^p(J_1, J_2)\right]^{1/p} \quad (12)$$

After that, Eq. (10) follows for any transverse distribution with a substitution

$$Q(\tau) \to Q_{\mathrm{eff}}(\tau) \equiv (q_{-2}^2 / q_{-1})Q_{\max}(\tau)$$

For the round Gaussian distribution, it yields $q_{-1} = 0.58$, $q_{-2} = 0.55$, $q_{-3} = 0.52$, and $q_{-2}^2 / q_{-1} = 0.52$.

For any real eigenvalue ν, Eq. (10) has two independent solutions, even and odd one. In general, these solutions tend to non-zero constants at the tails, $\lim(\bar{y}(\tau))_{\tau\to\infty} = \bar{y}(\infty)$, while their derivatives $\bar{y}'(\tau)$ tend to zero,

$$\bar{y}'(\tau) \cong \frac{Q_{\mathrm{eff}}(\tau)}{Q_{\mathrm{eff}}(0)}\cdot(a - b\tau), \quad (13)$$

with constants $a$ and $b = \nu\bar{y}(\infty)$ being determined by the eigenvalue ν. Any boundary condition would select a sequence of discrete eigenvalues. Note that the strong space charge and rigid beam approximations fail at the bunch tails. Namely, these assumptions are violated at that longitudinal offset, where the function $\bar{y}(\Psi)$ cannot be considered as slow function of the space charge phase Ψ (see Eqs. 5-9). This happens at $\tau = \tau_*$, where

$$\left| \frac{d}{d\Psi} \bar{y}'(\tau_*) \right| \cong \left| \bar{y}'(\tau_*) \right| \cdot$$

Using Eq. (5,13), this yields an equation for that model-break point $\tau_*$ at the bunch tail:

$$Q(\tau_*) = u(\tau_*) \,|\, Q'(\tau_*) \,|\, / \, Q(\tau_*) \,. \tag{14}$$

The individual particles, being essentially in coherent motion before that, go incoherently after that, so this point can be called as a decoherence point. The gradients in coherent motion smear out much faster after that point, than it would go according to Eq. (13). Thus, we are coming to the boundary condition:

$$\bar{y}'(\pm\tau_*) = 0 \,. \tag{15}$$

This boundary condition is identical to what would be required if there were a vertical potential barrier at the model-breaking point. This additional meaning of this boundary condition appears to be reasonable by itself. Indeed, setting that barrier at the model-breaking point makes the model applicable everywhere. At the same time, since it is set at that far tails, it almost does not change the collective dynamics of the bunch. The idea of model breaking implies that the right-hand side of Eq. (14) is defined up to a numerical factor ~ 1. However, since at the far tails the left-hand side of Eq. (14) is extremely fast function of its argument, the decoherence point $\tau_*$ is defined with rather good accuracy at strong space charge.

Eqs (10, 15) reduce the general problem of eigen-modes to a well-known mathematical boundary-value problem, similar to the single-dimensional Schrödinger equation. This problem is normal, so it has full orthonormal basis of the eigen-functions at the interval $(-\tau_*, \tau_*)$.

$$\int_{-\tau_*}^{\tau_*} \bar{y}_k(\tau) \bar{y}_m(\tau) \frac{d\tau}{u(\tau)} = \delta_{km} \cdot$$

As a consequence,

$$\sum_{m=0}^{\infty} \bar{y}_m(\tau) \bar{y}_m(s) = u(\tau) \delta(s - \tau) \cdot$$

At the bunch core, the $k$-th eigen-function $\bar{y}_k(t)$ behaves like $\sim \sin(k\tau/\sigma)$ or $\sim \cos(k\tau/\sigma)$, and the eigenvalues are estimated to be

$$\nu_k \cong k^2 \bar{Q}_s^2 / Q_{\rm eff}(0) << \bar{Q}_s \;, \tag{16}$$

which are similar to the values in the square well case.
For a weak head-tail case, the coherent tune shift and the coherent detuning are given as perturbations, by their diagonal matrix elements, similar to the analogous Quantum Mechanical results:

$$Q_w = \kappa \int_{-\infty}^{\infty} \int_{\tau}^{\infty} W(\tau - s) \exp(i\zeta(\tau - s)) \rho(s) \bar{y}_k(s) \bar{y}_k(\tau) u^{-1}(\tau) ds d\tau \,, \tag{17}$$

$$Q_d = \kappa \int_{-\infty}^{\infty} \int_{\tau}^{\infty} D(\tau - s) \rho(s) \bar{y}_k^2(\tau) u^{-1}(\tau) ds d\tau \,.$$

From here, it follows that a sum of all growth rates is zero:

$$\sum_{k=0}^{\infty} \mathrm{Im}\, Q_w = 0 \cdot \tag{18}$$

This statement sometimes is referred to as the growth rates sum theorem. Note also that the detuning wake does not introduce any growth rate, and every growth rate is proportional to the head-tail phase when this phase is small, similar to the conventional no-space-charge case.

For a short wake, $W(\tau) = -G\delta'(\tau)$, the growth rate can be expressed as

$$\mathrm{Im}\, Q_w = \kappa \rho_k \zeta\, G \,; \quad \rho_k \equiv \int_{-\infty}^{\infty} \rho(\tau) \bar{y}_k^2(\tau) u^{-1}(\tau) d\tau \,, \tag{19}$$

in agreement with the special result for a square well found in Ref. [6].

Growth rates (17) as functions of the head-tail phase for a Gaussian bunch and resistive wake $W(s) = W_0 / \sqrt{s}$ are presented at Fig.2.

## MODES FOR GAUSSIAN BUNCH

The Gaussian distribution in phase space,

$$f(\mathrm{v}, \tau) = \frac{N_b}{2\pi\, \sigma\, u} \exp(-\mathrm{v}^2 / 2u^2 - \tau^2 / 2\sigma^2) \,, \tag{20}$$

describes a thermal equilibrium of a bunch whose length is much shorter than the RF wavelength. Below, natural units for Eq. (10) for the Gaussian bunch are used. The distance τ is measured in units of the bunch length σ, and the eigenvalue $\nu$ - in units of $u^2 / \left(\sigma^2 Q_{\rm eff}(0)\right) = Q_s^2 / Q_{\rm eff}(0)$.

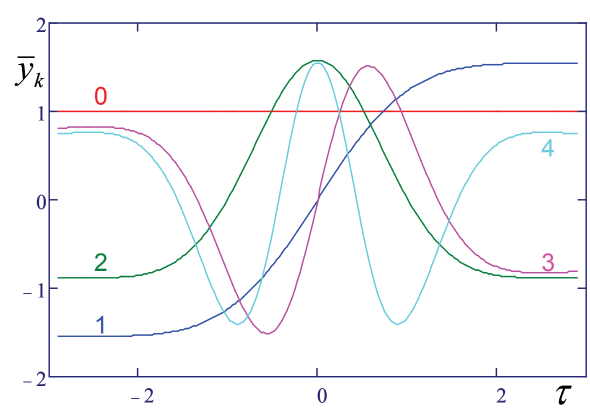


Figure 1: The first five eigenfunctions for the Gaussian beam at $\tau_* = 2.5$ (or $q$=60) as functions of the dimensionless distance along the bunch τ, Eq. (21). The eigenfunctions are identified by their mode numbers.

Then, the boundary-value problem of Eqs. (10, 15) is written as

$$\nu_k \bar{y} + \frac{d}{d\tau}\left(e^{\tau^2/2}\frac{d\bar{y}}{d\tau}\right) = 0;$$
$$\bar{y}'(\pm\tau_*) = 0 \; ; \qquad (21)$$
$$\tau_* = \sqrt{2\ln(q/\tau_*)}\; ; \; q \equiv Q_{\text{eff}}(0)/Q_s \, .$$

This equation is easily solved numerically. A list of first ten eigenvalues $\nu_k$ found for τ*=1.5, 2.0 and 2.5 (corresponding to *q*=5, 15 and 60) is presented in the Table 1.

Table 1: First ten eigenvalues $\nu_k$ of the Gaussian bunch (Eq. 21) for τ*=1.5, 2.0, 2.5 or *q*=5, 15 and 60, *q*>>2*k*

| τ* \ *k* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.5 | 0 | 1.2 | | | | | | | | |
| 2.0 | 0 | 0.78 | 4.0 | 9.2 | 17 | | | | | |
| 2.5 | 0 | 0.55 | 3.2 | 7.7 | 14 | 22 | 32 | 45 | 60 | 75 |

All these eigenvalues are limited as

$$k^2 - k/2 \le \nu_k \le k^2 + k/2\,; \quad k = 0,1,2... \qquad (22)$$

These numbers are only logarithmically sensitive to the space charge parameter $q >> 2k$. The first four eigenfunctions, normalized to the unit energy,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{y}_k^2(\tau)e^{-\tau^2/2}d\tau = 1\,, \qquad (23)$$

are shown in Fig. 1.

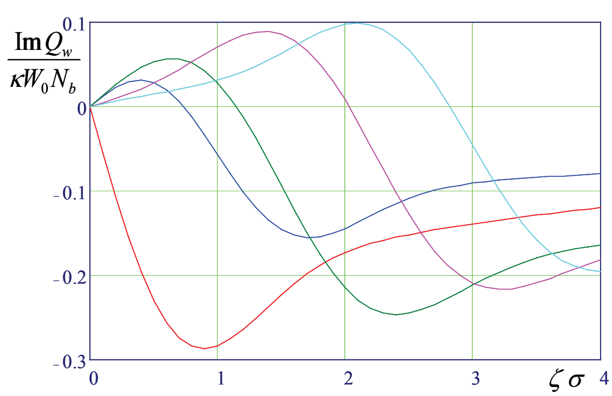


Figure 2: Coherent growth rates for the Gaussian bunch with the resistive wake $W(s) = W_0/\sqrt{s}$ as functions of the head-tail phase $\zeta\sigma$, for the lowest mode 0 (red), mode 1 (blue), 2 (green), 3 (magenta) and 4 (cyan).

With the modes of the Gaussian bunch found, the coherent growth rates can be calculated according to Eq. (17), see Fig. 2.

## LANDAU DAMPING

Landau damping is a mechanism of dissipation of coherent motion due to transfer of its energy into incoherent motion of resonant particles. This energy transfer is impossible at the bunch center, where the coherent and incoherent frequencies are strongly separated, but it gets effective at the bunch tails, where the local incoherent space charge tune shift becomes small enough. For a given location τ, this energy transfer occurs for those particles whose velocities $v_i$ and individual space charge tune shifts $Q_i(\tau)$ relate as

$$Q_i(\tau) \cong v_i/\Delta\tau = |v_i Q'(\tau)|/Q(\tau) = v_i\tau\,. \qquad (24)$$

The individual local space charge tune shift in Eq. (24) is a function of the two transverse actions:

$$Q_i(\tau) \equiv Q(J_1, J_2, \tau)\,. \qquad (25)$$

Using that the particle's longitudinal offset and velocity relate to its longitudinal action,

$$J_\parallel = \frac{\tau^2}{2} + \frac{v^2}{2}\,, \qquad (26)$$

this defines at the given longitudinal position τ a 2D Landau surface in the space of three actions $J_1, J_2, J_\parallel$:

$$J_\parallel = \frac{\tau^2}{2} + \frac{Q^2(\tau, J_1, J_2)}{2\tau^2}\,. \qquad (27)$$

After passing its 'Landau point' (24), the particle gets the variable part of the coherent amplitude

$$\tilde{y}(\tau) \cong \bar{y}'(\tau)Q(\tau)/|Q'(\tau)| \qquad (28)$$

as its incoherent amplitude. For the Gaussian distribution

$$\tilde{y}(\tau) = \bar{y}'(\tau)/\tau\,. \qquad (28a)$$

After *M* times of passing its Landau point, the particle gets its individual amplitude excited by

$$\Delta y_i(M) = \tilde{y}\sum_{m=0}^{M-1}e^{im\psi} = \tilde{y}e^{iM\psi/2}\frac{\sin(M\psi/2)}{\sin(\psi/2)}\,,$$

where $\psi$ is the space charge phase advance Ψ per synchrotron period $\theta_s$, see Eq. (5). The entire Landau energy transfer for the bunch after *M*>>1 turns can be expressed as

$$\Delta E(M) = 4\int d\mathbf{J} f(\mathbf{J})\tilde{y}^2\frac{\sin^2(M\psi/2)}{\sin^2(\psi/2)}\,,$$

where ***J*** is 3D vector of the three actions, and the 3D integral over actions has to be understood as

$$\int d\mathbf{J}(...) = \int_0^\infty dJ_1\int_0^\infty dJ_2\int_0^\infty d\tau\frac{\partial J_\parallel}{\partial\tau}(...),\cdot$$

The contributions from particle entering and leaving the tails are assumed equal in magnitude but with random relative phase. The power of the Landau energy transfer is calculated as

$$\Delta\dot{E} = \frac{d\Delta E(M)}{T_s dM} = 4Q_s\int d\mathbf{J} f(\mathbf{J})\tilde{y}^2\delta_P(\psi)\,,$$
$$\delta_P(\psi) \equiv \sum_n \delta(\psi - 2\pi n)\,.$$

Since $\psi$>>1, the sum over many resonance lines *n* can be approximated as an integral, $\delta_P(\psi) \to 1/(2\pi)$, yielding

$$\Delta\dot{E} = \frac{2Q_s}{\pi}\int d\mathbf{J} f(\mathbf{J})\tilde{y}^2\,. \qquad (29)$$

From here, the Landau damping rate $\Lambda_k = \Delta\dot{E}/2E_k$ follows for any kind of bunch 3D distribution, with $E_k$ as the energy number:

$$\Lambda_k = \frac{Q_s}{\pi E_k}\int d\mathbf{J} f(\mathbf{J})\tilde{y}_k^2\,; \quad E_k = N_b^{-1}\int_{-\infty}^{\infty}\bar{y}_k^2(\tau)\rho(\tau)d\tau \qquad (30)$$

For the longitudinal Gaussian distribution, assuming the eigenfunctions normalized by the unit energy $E_k$, as they were calculated in the previous section, it yields

$$\Lambda_k = \frac{Q_s}{\pi \tau_*^2} \int d\mathbf{J} f(\mathbf{J}) y_k'^2 . \quad (30a)$$

According to Eq. (21), asymptotically

$$y' \cong b \cdot (\tau - \tau_*) \frac{Q_{\text{eff}}(\tau)}{Q_{\text{eff}}(0)} = b \cdot (\tau - \tau_*) \exp(-\tau^2 / 2), \quad (31)$$

where the asymptotic parameter $b$ can be calculated for every eigenmode, $b=b_k$. For the modes of the Gaussian bunch, numerically found squares of these parameters are presented in the Table 2; the damping rate is seen as extremely sensitive to the mode number. Since the number of the lowest potentially unstable mode is about the chromatic head-tail phase, an increase of the chromaticity has to be a powerful tool for the beam stabilization.

Table 2: Mode asymptotic parameters $b_k^2$ for Gaussian bunch

| $\tau_*\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.5 | 0 | 2.5 | | | | | | | | |
| 2.0 | 0 | 1.3 | 15 | 64 | 160 | | | | | |
| 2.5 | 0 | 0.85 | 7.5 | 40 | 105 | 260 | 500 | 1060 | 1700 | 2300 |

For the Gaussian bunch, the Landau damping rate is calculated as

$$\Lambda_k = 1.5 b_k^2 Q_s / q^3 ; \quad q = \frac{Q_{\text{eff}}(0)}{Q_s} . \quad (32)$$

Note that the synchrotron tune and the space charge tune shift enter in high powers in Eq. (32).

Eqs (30, 32) give Landau damping, assuming ideally linear lattice and longitudinal RF force. That is why that kind of Landau damping can be called intrinsic. Although the method of calculation is general, the specific results assume small coherent tune shift between the two neighbor modes, $Q_w << Q_s^2 / Q_{\text{eff}}(0)$. In particular, independence of the Landau damping rate of the chromaticity should not be expected for larger wake terms.

Similar method was applied to calculate Landau damping rate caused by the lattice nonlinearity; the results can be found in Ref. [10].

## SUMMARY

In this paper, a theory of head-tail modes is presented for space charge tune shift significantly exceeding the synchrotron tune, which is rather typical case for hadron machines. A general equation for the modes is derived for any ratio of the synchrotron tune and the wake-related coherent tune shift. Without the wake term, this is a 2-nd order self-adjoint ordinary differential equation, known to have full orthonormal basis of the eigenfunctions. The spectrum of this equation is discussed in general and solutions for the Gaussian bunch are presented in detail. Intrinsic Landau damping of the space charge modes is calculated. Lattice nonlinearity as a source of additional Landau damping is taken into account in more detailed Ref. [10], where also specific behavior of transverse mode coupling instability for the space charge modes is discussed.

The presented theory needs to be compared with simulations and measurements. The author hopes this will happen in a near future.

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