# SMALL-SIGNAL THEORY OF SPACE-CHARGE WAVES ON RELATIVISTIC ELLIPTIC ELECTRON BEAMS

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*Abstract*

This paper reports on results of a small-signal analysis of space-charge waves on a relativistic elliptic electron beam immersed in a strong axial magnet field in a perfectly-conducting tunnel with an elliptic cross section. A dispersion relation for the space-charge waves is derived analytically. A computer code, Elliptic Beam Small Signal (EBSS), is developed and used in studies of the dispersion characteristics of fast- and slow-space-charge waves on relativistic elliptic electron beams. Applications of the theory in elliptic-beam klystrons are discussed.

## INTRODUCTION

Elliptic electron beams have a variety of applications in coherent radiation sources and particle accelerators. Recently, there have been vigorous theoretical and experimental studies of elliptic beam systems, including elliptic electron beam sources [1,2], elliptic beam focusing systems [3-7], sheet-beam klystrons [8,9] and traveling wave tubes (TWTs) [10,11]. In particular, relativistic elliptic electron beams have applications in the research and development of a new class of elliptic- or sheet-beam klystrons which have the potential to outperform conventional klystrons in terms of power, efficiency, and operating voltage. They have lower space-charge energies which allow higher currents to be used in beams and higher power devices. Furthermore, the elliptic cross section allows more efficient coupling to radio frequency structures than circular beams. Cavity spacing in an elliptic-beam klystron is dependent on the beat wavelength of space charge waves on the elliptic beam. While space-charge waves on circular beams have been studied extensively [12-14], there has been little theoretical study of space-charge waves on elliptic electron beams until this paper.

## THEORY

In this paper, we report on results of a small-signal analysis of space-charge waves on a relativistic elliptic electron beam in a perfectly-conducting tunnel with an elliptical cross section. We consider a relativistic elliptic electron beam propagating in an infinitely strong axial magnetic field inside a perfectly conducting tunnel whose inner surface $\xi = \xi_t(\eta)$ corresponds to an elliptic pipe which approximates an equipotential surface [6] produced by the equilibrium beam charge distribution. The equilibrium electron density and flow velocity profiles are assumed to be the following [15]:

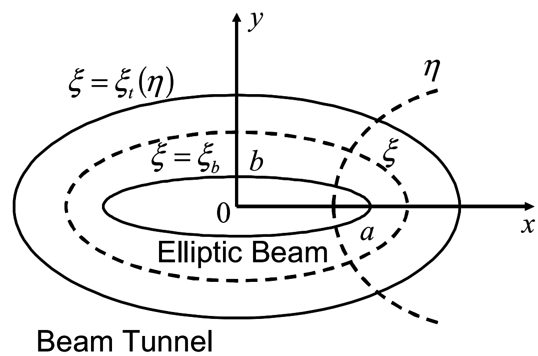


Figure 1: Cross section of a relativistic elliptic electron beam system.

$$n(\xi) = \begin{cases} n_b, & \xi < \xi_b, \\ 0, & \xi > \xi_b, \end{cases} \tag{1}$$

and

$$\mathbf{V} = \beta_b c \hat{\mathbf{e}}_z, \tag{2}$$

where $\xi$ is the radial elliptic coordinate in the elliptic-cylindrical coordinate system defined as $x = f \cosh\xi \cos\eta$ and $y = f \sinh\xi \sin\eta$ with $f = \sqrt{a^2 - b^2}$, $\xi = \xi_b = \tanh^{-1}(b/a)$ is the beam boundary, $c$ is the speed of light in vacuum, and $\beta_b$ is a constant. Equations (1) and (2) fully specify the equilibrium state of a relativistic elliptic electron beam. We introduce a small sinusoidal perturbation into the system so that the general form of a field variable $\psi$ is

$$\psi = \psi_0(x, y) + \delta\psi(x, y) e^{i(kz - \omega t)}, \tag{3}$$

where $\psi_0(x, y)$ denotes the equilibrium field variable and $\delta\psi(x, y)$ denotes the amplitude of the perturbation. From the linearized cold-fluid equations, it is readily shown that the eigenvalue equation is

$$\frac{2}{f^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\delta E_z = p^2\left[1 - \frac{e^2 n_b}{\varepsilon_0 m_e \gamma^3 (\omega - k\beta_b c)^2}\right]\delta E_z \tag{4}$$

where $p^2 = k^2 - \omega^2 / c^2$, $-e$ is the electron charge, $m_e$ is the electron mass, $\varepsilon_0$ is the vacuum permittivity, and $\gamma = (1-\beta_b^2)^{-1/2}$ is the relativistic factor. The boundary conditions for the eigenvalue equation are

$$\delta E_z\big|_{\xi=\xi_t(\eta)} = 0 \,, \tag{5}$$

$$\delta E_z\big|_{\xi=\xi_b^-} = \delta E_z\big|_{\xi=\xi_b^+} \,, \tag{6}$$

$$\left.\frac{\partial \delta E_z}{\partial \xi}\right|_{\xi=\xi_b^+} - \left.\frac{\partial \delta E_z}{\partial \xi}\right|_{\xi=\xi_b^-} = 0 \,. \tag{7}$$

Solving Eq. (4) by separation of variables under the boundary condition in Eqs. (5) and (6) and making use of the fact that $\delta E_z$ must be finite, we obtain

$$\delta E_z = \begin{cases} \sum_{n=0}^{\infty} A_n \,\mathrm{Ce}_{2n}(\xi,-\frac{p_1^2 f^2}{4})\,\mathrm{ce}_{2n}(\eta,-\frac{p_1^2 f^2}{4}), \xi < \xi_b \\ \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} B_l \Big[\delta_{nl}\,\mathrm{Ce}_{2n}(\xi,-\frac{p^2 f^2}{4}) \\ + R_{nl}\,\mathrm{Fek}_{2n}(\xi,-\frac{p^2 f^2}{4})\Big]\mathrm{ce}_{2n}(\eta,-\frac{p^2 f^2}{4}), \xi > \xi_b \end{cases} \tag{8}$$

In Eq. (8), $A_n$ and $B_n$ are coefficients; $\delta_{nl}$ is the Kronecker delta; $\mathrm{ce}_{2n}$ is the even angular Mathieu function of order $2n$, and $\mathrm{ce}_{2n}$ and $\mathrm{FeK}_{2n}$ are the first and second radial Mathieu functions of order $2n$ [16];

$$p_1^2 = p^2\left[1 - \frac{e^2 n_b}{\varepsilon_0 m_e \gamma^3 (\omega - k\beta_b c)^2}\right]; \tag{9}$$

$$\mathbf{R} \equiv \mathbf{T}^{-1}\mathbf{S} \,, \tag{10}$$

where the elements of **S** and **T** are

$$\begin{aligned} S_{mn} &= \int_0^{2\pi} \cos(2m\eta)\,\mathrm{Ce}_{2n}(\xi_t,-\frac{p^2 f^2}{4}) \\ &\times\ \mathrm{ce}_{2n}(\eta,-\frac{p^2 f^2}{4})\,d\eta \end{aligned} \tag{11}$$

$$\begin{aligned} T_{mn} &= -\int_0^{2\pi} \cos(2m\eta)\,\mathrm{Fek}_{2n}(\xi_t,-\frac{p^2 f^2}{4}) \\ &\times\ \mathrm{ce}_{2n}(\eta,-\frac{p^2 f^2}{4})\,d\eta \end{aligned} \tag{12}$$

Matching the boundary condition in Eq. (7) yields the dispersion relation

$$\det \mathbf{D}(\omega, k) = 0 \,. \tag{13}$$

In Eq. (13),

$$\begin{aligned} D_{mn} &= \int_0^{2\pi} \cos(2m\eta) \sum_{l=0}^{\infty} \mathrm{ce}_{2l}(\eta,-\frac{p^2 f^2}{4}) \\ &\times \Big[\mathrm{Ce}'_{2l}(\xi_b,-\frac{p^2 f^2}{4})V_{ln} \\ &+\ \mathrm{Fek}'_{2l}(\xi_b,-\frac{p^2 f^2}{4})Q_{ln}\Big]d\eta \\ &-\ \int_0^{2\pi} \cos(2m\eta)\,\mathrm{Ce}'_{2n}(\xi_b,-\frac{p_1^2 f^2}{4}) \\ &\times\ \mathrm{ce}_{2n}(\eta,-\frac{p_1^2 f^2}{4})\,d\eta \end{aligned} \tag{14}$$

$$\mathbf{Q} \equiv \mathbf{R}\mathbf{V} \equiv \mathbf{R}\mathbf{W}^{-1}\mathbf{U} \,, \tag{15}$$

where the prime denotes derivative with respect to $\xi$ and the elements of **U** and **W** are

$$U_{mn} = \int_0^{2\pi} \cos(2m\eta)\,\mathrm{Ce}_{2n}(\xi_b,-\frac{p_1^2 f^2}{4})\,\mathrm{ce}_{2n}(\eta,-\frac{p_1^2 f^2}{4})\,d\eta \,, \tag{16}$$

$$\begin{aligned} W_{mn} &= \int_0^{2\pi} \cos(2m\eta) \sum_{l=0}^{\infty} \mathrm{ce}_{2l}(\eta,-\frac{p_1^2 f^2}{4}) \\ &\times \Big[\mathrm{Ce}_{2l}(\xi_b,-\frac{p^2 f^2}{4})\delta_{ln} + \mathrm{Fek}_{2l}(\xi_b,-\frac{p^2 f^2}{4})R_{ln}\Big]d\eta \end{aligned} \tag{17}$$

In the limit of a circular beam ($b \to a$) in a circular pipe of radius $R$, we find that the angular Mathieu functions become sinusoids and that the radial Mathieu functions become modified Bessel functions scaled by a constant, e.g., $\mathrm{Fek}_0(\xi_b,-\frac{p^2 f^2}{4})$ becomes $-K_0(pa)/\sqrt{2\pi}$, where $a$ is the beam radius in cylindrical coordinates. The matrix **D** becomes diagonal, and Eq. (13) reduces to the known dispersion equation for a solid circular beam in a circular pipe [13], i.e.,

$$p_1 \frac{I_1(p_1 R)}{I_0(p_1 R)} = p\left(\frac{K_0(pR)I_1(pa) + K_1(pa)I_0(pR)}{K_0(pR)I_0(pa) - K_0(pa)I_0(pR)}\right), \tag{16}$$

where $I_n$ and $K_n$ are the first- and second-kind modified Bessel functions of order $n$.

We have developed an Elliptic Beam Small Signal (EBSS) code for computing the roots of $\det \mathbf{D}(\omega, k) = 0$ for given system parameters. The EBSS makes use of Python and the SciPy library for the angular Mathieu functions, Bessel functions, and numerical integration. In the EBSS code, we approximate the radial Mathieu functions using a finite number of terms from the Bessel function series described in [16]. To benchmark this approximation, we have computed tables of values of the radial Mathieu functions using EBSS and have compared them to those generated by Mathematica, obtaining identical results. We have also benchmarked the EBSS code against known numerical results for the case of a

Table 1: Parameters of a Relativistic Elliptic Electron Beam

| | |
|---|---|
| Current | 111 A |
| Voltage | 120 kV |
| Beam semi major axis $a$ | 0.5 cm |
| Beam semi minor axis $b$ | 0.05 cm |
| Beam tunnel semi major axis $a_t$ | 0.8 cm |
| Beam tunnel semi minor axis $b_t$ | 0.6 cm |

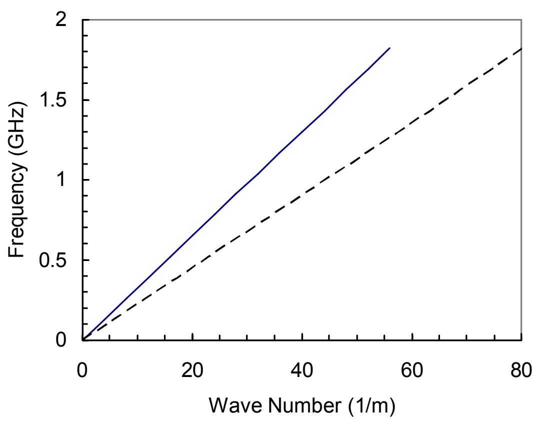


Figure 2: Dispersion characteristics obtained from Eq. (13) for the parameters listed in Table 1. The solid curve is for the fast-space-charge waves, whereas the dashed curve is for the slow-space-charge waves.

circular beam in a circular pipe, obtaining identical results. Because the EBSS code also truncates the matrices to a finite size, we have tested the convergence of the results for elliptic beams. Typically, the results converge to within 0.1% for matrices of size $3\times3$.

Figure 2 shows a plot of the wave dispersion characteristics for a 10:1 elliptic beam with the parameters listed in Table 1. The 10:1 elliptic beam is relevant to the development of a 10 MW, L-Band sheet- or elliptic-beam klystron for the International Linear Collider (ILC) [17]. Both fast- and slow-space-charge waves exist, as in the case of circular beams. For the frequencies plotted, the long-wavelength limit is applicable, which explains why each curve is linear. The difference $\Delta k = k_s - k_f$ between the wave numbers of the slow- and fast-space-charge waves for a given frequency is inversely proportional to the beat wavelength of the system which determines the spacing of rf cavities when constructing an elliptic-beam klystron. For a 1.3 GHz elliptic-beam klystron, the spacing between the input and second rf cavities is $L = \pi/\Delta k = 17.9$ cm.

## CONCLUSIONS

A theory of small-signal space-charge waves on relativistic elliptic electron beams has been presented. A dispersion relation has been derived. A computer code has been developed and used for numerical studies of dispersion characteristics. Applications of the theory in elliptic-beam klystrons have been discussed.

## ACKNOWLEDGMENTS

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