

Advances in Impedance Theory

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Outline of the talk

- Introduction
- Geometric impedance of tapered transitions:
 - small angle approximation in the impedance theory
 - elliptical tapers
 - frequency dependence of the impedance for gradual tapers
- Optical approximation—the high frequency limit of the impedance
- Using parabolic equation in the impedance theory
- Resistive wall impedance of inserts
- Conclusion and outlook

There are some other important contributions to the theory of impedance left outside of this talk: CSR impedance, resistive wall impedance in the limit of large skin depth, time dependent wakefields, etc.

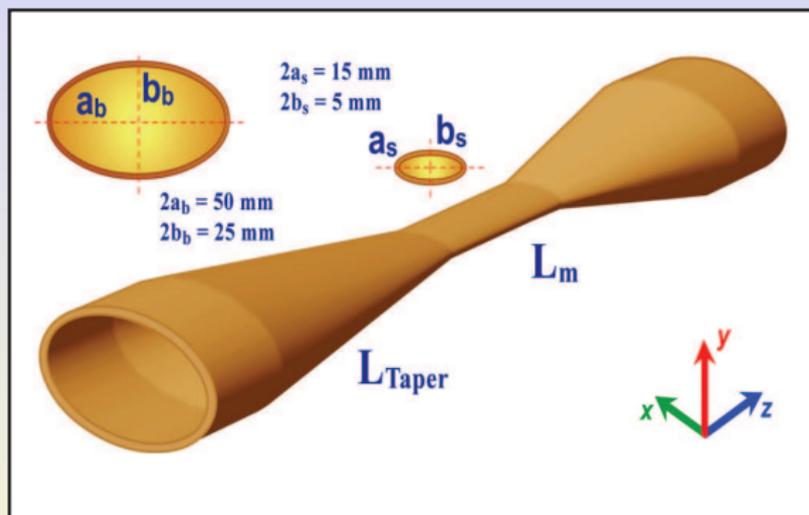
Do we need an analytical theory of impedance?

A remarkable progress over the last decade in development of computer codes significantly advanced our capabilities in calculation of wakefields and impedances for accelerators. Do we need a theory?

- The theory allows a quick evaluation of the impedance and often gives scalings of how impedance depends on various parameters of the problem. Codes require more effort (and money).
- A new code is usually benchmarked against the theory
- A good theory can simplify *numerical approach* to the problem and allows to advance to a region of parameter where codes are either too slow or require large computer resources

Small-angle tapers: motivation

Long, small-angle tapers which are often used to minimize the abruptness of vacuum chamber transitions.



An example from the NSLS-II design (A. Blednykh): $\sigma_z = 4.5 \text{ mm}$,
 $L_{\text{Taper}} = 18 \text{ cm}$.

Tapered axisymmetric transitions

Yokoya (1990) derived both the longitudinal and transverse impedances of *a*) a small-angle axisymmetric transition in *b*) the low-frequency approximation

$$Z_{\perp} = -\frac{iZ_0}{2\pi} \int dz \left(\frac{a'}{a} \right)^2 \sim \frac{1}{L}$$

$$Z_0 = 4\pi/c = 377 \text{ Ohm}$$

$a = a(z)$ the radius of the pipe, $|a'| \ll 1$

Podobedov and Krinsky (2006) found higher-order correction terms in a' .

Conical taper and higher-order terms



Conical transition with angle θ connects two pipes of radii a_1 and a_2 ($a_2 > a_1$):

$$Z_{\perp} = -\frac{iZ_0}{\pi a_{av}} \frac{\epsilon \tan \theta}{1 - \epsilon^2}$$

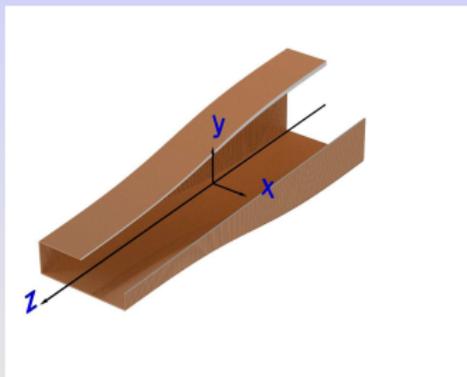
where $a_{av} = (a_1 + a_2)/2$ and $\epsilon = (a_2 - a_1)/(a_2 + a_1)$.

Podobedov and Krinsky (2006) found a leading order correction for the conical taper

$$Z_{\perp} = -\frac{iZ_0}{\pi a_{av}} \frac{\epsilon \tan \theta}{1 - \epsilon^2} \left(1 - \frac{0.18}{\epsilon} \tan \theta \right)$$

Using several fitting parameters, they also found interpolation formulas for the impedance of the taper that provide an excellent approximation for $0 < \theta < \pi/2$ and $0 < \epsilon < 1$.

Non-round tapers and large aspect ratio transitions



Wide transitions are often used in practice.

For a non-axisymmetric transitions with reflection symmetry there are two dipole impedances $Z_{\perp y}$, $Z_{\perp x}$ and a quadrupole one Z_Q .

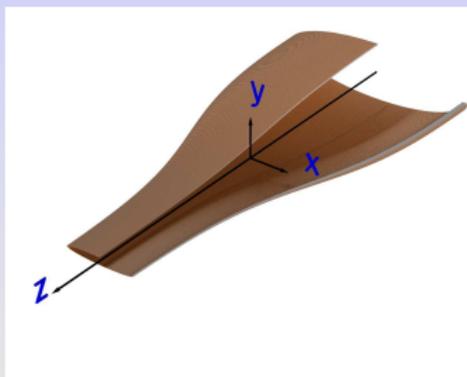
It was found [Stupakov (1996,1997,2007)] that the impedance of a rectangular transition with a large aspect ratio is $\sim w/g$ times larger than that of a round one with the same length L

$$Z_{\perp y} = -\frac{i}{4}wZ_0 \int dz \frac{(g')^2}{g^3}$$

w is the (constant) width in the x direction

$g = g(z)$ is the (varying) gap of the taper in the y direction

Tapers with elliptic cross section



Podobedov and Krinsky (2007) calculated the impedance of a taper with confocal elliptical cross sections and confirmed a similar scaling with the width of the ellipse in the limit of large aspect ratios.

$$Z_{\perp y} = -\frac{i\pi}{16} Z_0 \int dz \frac{(\rho')^2}{\rho^3}$$

$$\rho(z) = \tanh^{-1}[b(z)/a(z)]$$

a and b are the major (horizontal) and minor (vertical) semiaxes of the elliptical cross section.

In the limit of large aspect ratio, $a \gg b$, it follows from this formula that $Z_{\perp y} \propto a$.

Conditions of applicability—geometric

For the round geometry $a \ll L$, or a small angle taper, $\theta \sim a/L \ll 1$.

For a large aspect ratio rectangular transition, in addition to requirement of small angle, it is required that $w \ll L$; similarly $a \ll L$ for the elliptic case, [Podobedov and Krinsky (2007)].

For a very wide taper, $g \ll L \ll w$ the vertical impedance as a function of width w saturates at $w \sim L$ and does not increase with the width when w becomes larger than L [Krinsky (2005)].

Conditions of applicability—low-frequency

For the round geometry, Yokoya's formula is valid for

$$k \ll L/a^2$$

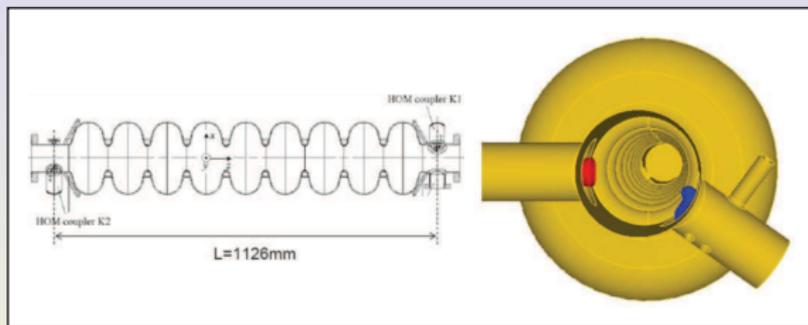
$$k = \omega/c.$$

For a wide rectangular transition the inductive regime is valid for $k \ll L/w^2$ [Stupakov (2001)]. Impedance at higher frequencies exhibits a more complicated dependence on ω .

Numerical simulations [Podobedov and Krinsky (2007)] partially confirmed the theoretical model at higher frequencies, but also showed some discrepancies.

High-frequency impedance computation: motivation

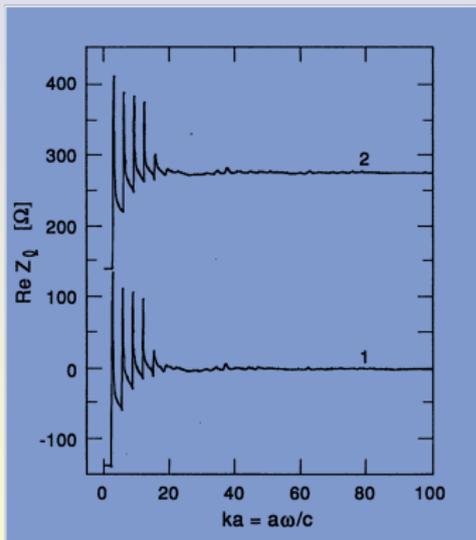
With the quest toward high-current high-brightness beams, the bunch length becomes extremely short. Examples: for ILC $\sigma_z = 300$ microns, for LCLS $\sigma_z = 20$ microns. One is interested in impedances at frequencies with $\lambda \sim 2\pi\sigma_z$.



A TESLA-type superconducting cavity with the fundamental and higher mode asymmetric couplers. The couplers introduce wakefields that kick the beam in the transverse direction and can degrade the emittance [K. Bane et al., paper TUPP019, EPAC08].

High-frequency approximation in electrodynamics

- The limit of very high frequency in classical electrodynamics is *geometric optics*. Light propagates (in vacuum) along straight lines and is reflected by metal surfaces. Can one apply this approximation in the theory of impedances?



Example: the longitudinal impedance of a step transition does not depend on ω at high frequencies (Heifets&Kheifets, 1991). For a step-out and an iris ($Z_0 = 377 \text{ Ohm}$)

$$Z_{||} = \frac{Z_0}{\pi} \ln \frac{a}{b}.$$

General *optical* theory of impedance

A comprehensive impedance theory in optical approximations:
Bane, Stupakov and Zagorodnov (2007).

Main features of the theory

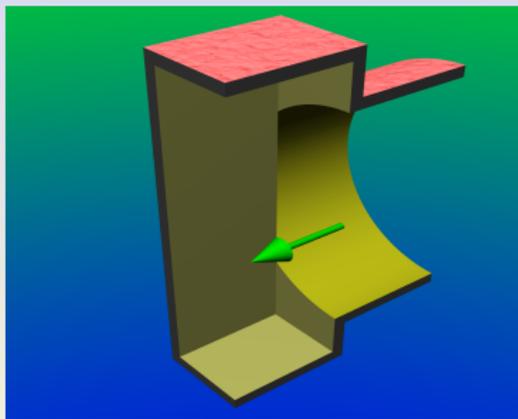
- Applicable for arbitrary 3D geometry
- Allows to compute both longitudinal and transverse impedances (through the Panofsky-Wenzel theorem $\mathbf{Z}_{\perp} = \frac{c}{\omega} \nabla_{\mathbf{r}_2} Z_{\parallel}$)
- The longitudinal impedance is purely resistive and does not depend on frequency

The optical theory—example

S_A – cross section of incoming pipe

S_B – cross section of outgoing pipe

S_{ap} is the minimal cross-section of the transition



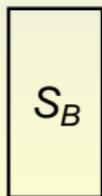
The longitudinal impedance does not depend on frequency

$$Z_{\parallel}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi c} I(\mathbf{r}_1, \mathbf{r}_2).$$

Wakefield is proportional to delta function of s

$$w_{\parallel}(\mathbf{r}_1, \mathbf{r}_2, s) = \frac{1}{2\pi} \delta(s) I(\mathbf{r}_1, \mathbf{r}_2).$$

where $\mathbf{r}_1 = (x_1, y_1)$ – 2D position of the leading point charge
 $\mathbf{r}_2 = (x_2, y_2)$ – 2D position of the trailing point charge,



Result of the optical theory

The factor I

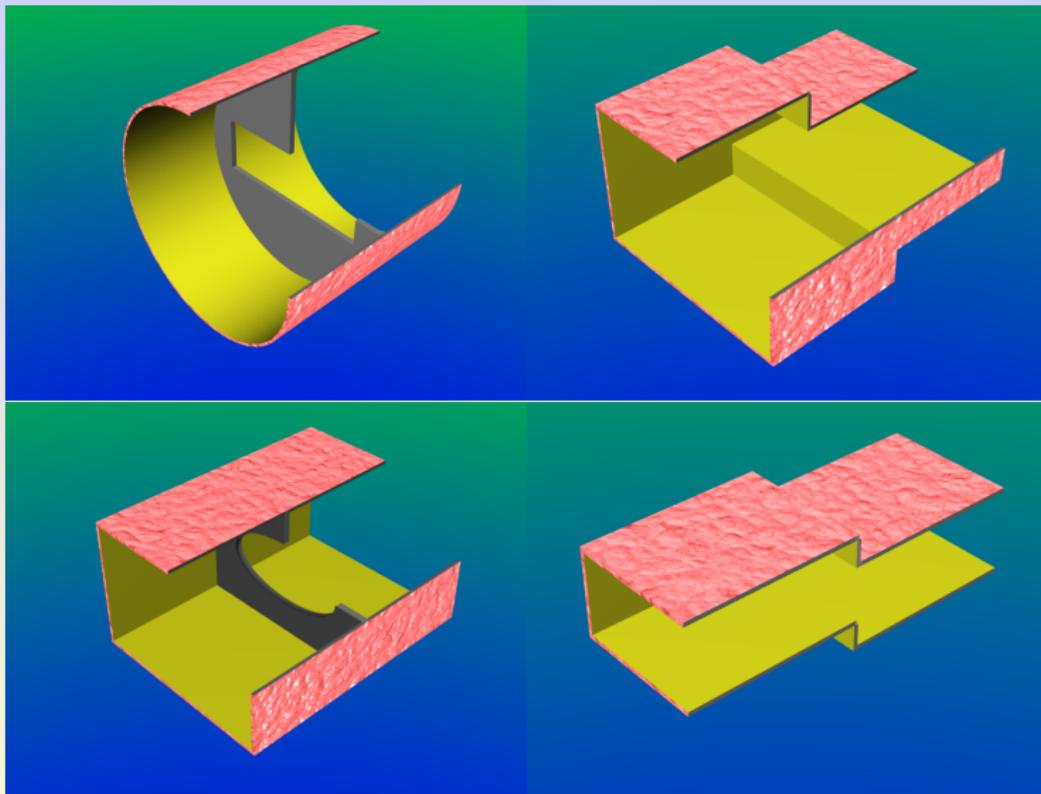
$$I = \int_{S_B} \nabla \phi_{1,B}(\mathbf{r}) \cdot \nabla \phi_{2,B}(\mathbf{r}) dS - \int_{S_{ap}} \nabla \phi_{1,A}(\mathbf{r}) \cdot \nabla \phi_{2,B}(\mathbf{r}) dS .$$

with $\nabla = \hat{\mathbf{x}} \partial / \partial x + \hat{\mathbf{y}} \partial / \partial y$. The potential ϕ satisfies Poisson's equation with delta functions on the RHS:

$$\nabla^2 \phi_{1,B}(\mathbf{r}) = -4\pi \delta(\mathbf{r} - \mathbf{r}_1) , \quad \nabla^2 \phi_{2,B}(\mathbf{r}) = -4\pi \delta(\mathbf{r} - \mathbf{r}_2) ,$$

with boundary conditions $\phi_{1,B} = \phi_{2,B} = 0$ on the wall of pipe B (and similar for $\phi_{1,A}$).

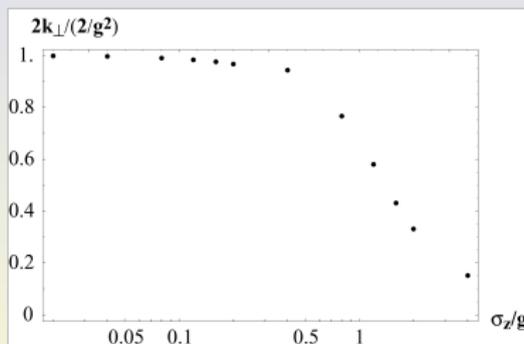
Various geometries treated in optical approximation



Comparison with numerical simulations

Results of the optical theory are compared with ECHO, a 3D, time-domain finite difference program that calculates wakefields of an ultra-relativistic bunch (I. Zagorodnov).

Test example: the transverse kick factor for a thin, round iris of radius g in a large beam pipe.



Kick factor vs bunch length

Parabolic equation for the electromagnetic field

Diffraction phenomena lie beyond the limits of the optical approximation. They can be accounted for in a simplified treatment based on a so called *parabolic equation*.

The parabolic equation in the diffraction theory was proposed many years ago [Leontovich and Fock, (1946)]. It is also a standard approximation in the FEL theory; it was applied to the beam radiation problems in a toroidal waveguide [Stupakov and Kotelnikov (2003), Agoh and Yokoya (2004)] and in free space [Geloni et al. (2005)].

Applicability of PE in the high-frequency limit of the impedance is based on the observation that in this case the main contribution to the impedance comes from the electromagnetic waves that propagate at small-angles to the axis of the pipe [Stupakov (2006)].

Parabolic equation for electromagnetic field

One introduces the *envelope* part of the electromagnetic field

$$\hat{\mathbf{E}}(x, y, z, \omega) = \int dt e^{i\omega t - ikz} \mathbf{E}(x, y, z, t).$$

The transverse (with respect to the direction of motion of the beam) component $\hat{\mathbf{E}}_{\perp}$, satisfies PE

$$\frac{\partial}{\partial z} \hat{\mathbf{E}}_{\perp} = \frac{i}{2k} \left(\nabla_{\perp}^2 \hat{\mathbf{E}}_{\perp} - \frac{4\pi}{c} \nabla_{\perp} \hat{j}_z \right)$$

\hat{j}_z is the Fourier transformed projection of the beam current in the direction z .

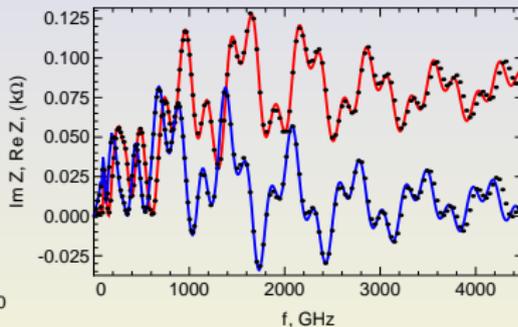
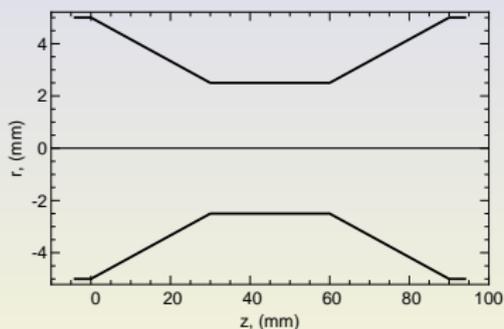
The longitudinal electric field \hat{E}_z can be expressed through $\text{div} \hat{\mathbf{E}}_{\perp}$.

PE eliminates the small wavelength $2\pi/k$ from the problem. The longitudinal scale L in that equation is of the order of $1/ka^2$, where a is the transverse size of the problem. As a result, the numerical solution of PE requires coarser spatial meshes.

Conical collimator example

The collimator has two tapered transitions of length 30 mm from radius of 5 mm to radius of 2.5 mm. It also has a central part (2.5 mm radius) of length of 30 mm.

The result is compared with the ECHO code.



At $\omega \rightarrow \infty$, $\text{Re } Z = (Z_0/\pi) \ln(a_{\text{max}}/a_{\text{min}}) = 83 \text{ Ohm}$, in agreement with the optical theory.

Impedance of resistive wall inserts



A pipe with a high wall conductivity (perfect conductor) has a short resistive insert of length L_i .

A naive approach to the case of an insert would be to multiply the impedance per unit length of an infinite pipe by the length of the insert L_i . This is true in the limit of low frequencies, $ks_0 \ll 1$, where $s_0 = (2a^2/Z_0\sigma)^{1/3}$ (σ is the wall conductivity, a is the pipe radius) [Stupakov (2005)].

Impedance of resistive wall inserts

At high frequencies, when $ks_0 \gg 1$, the detailed analysis [Krinsky et al, (2004)] reveals several regimes in this problem.

- One can use long-pipe approximation for the impedance of long inserts, when $L_i \gg ka^2$
- In an intermediate regime when $a^2/s_0^3 k^2 \ll L_i \ll ka^2$ the impedance does not depend on conductivity and is equal to the twice the diffraction impedance of a pill-box cavity of length L_i

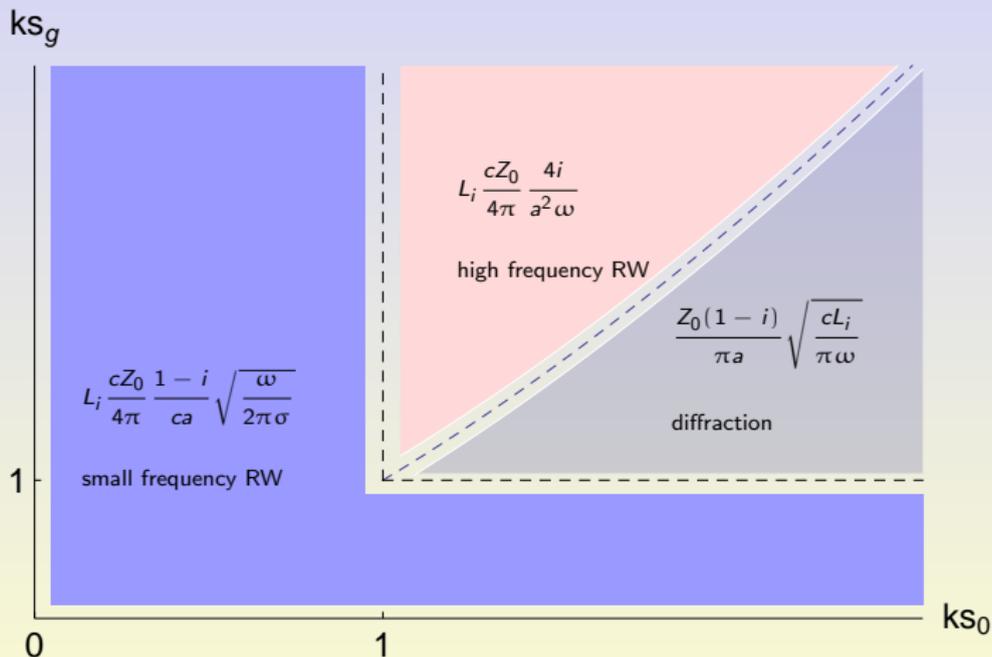
$$Z_{\parallel} = \frac{Z_0(1-i)}{\pi a} \sqrt{\frac{cL_i}{\pi\omega}}$$

- For very short inserts, $a^2/s_0^3 k^2 \gg L_i$,

$$Z_{\parallel} = L_i \frac{cZ_0}{4\pi} \frac{1-i}{ca} \sqrt{\frac{\omega}{2\pi\sigma}}$$

Impedance of resistive wall inserts

$$s_g = \sqrt{\frac{L_i}{2\sigma Z_0}}$$



Conclusion and outlook

- We have a good understanding of the impedance at low frequencies for gradual tapers and collimators including large aspect ratio geometries. Further research is needed to generalize and validate these results in the region of high frequencies.
- Optical theory gives us a simplified treatment of the high-frequency limit of the impedance for many practically important 3D geometries. It can be improved by usage of the parabolic equation approximation. A combination of these methods with existing computational should make feasible calculation of the exact Green functions (wake of a point charge) for many geometries.
- A complete theory of the RW impedance of round inserts is now available. A study of nonaxisymmetric geometries is desirable.