Gravitational Instability of a Nonrotating Galaxy

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Is a galaxy stable under its own gravitational self-forces?
Attempt to apply accelerator physics analysis to address this question.
Replace a beam for a galaxy:

Beam

- Consider a collection of charged particles in a beam
- Force between particles = electromagnetic
- Beam becomes unstable when its charge is too high

Galaxy

- Consider a collection of stars in a galaxy
- Force between stars = gravitation
- Galaxy becomes unstable when its mass is too large

Replace beam by galaxy, replace particles by stars,
replace wake fields by gravity.

Recall history: 19-century astrophysics has profound impact on 20/21-century accelerator physics. Two notable examples:

Nonlinear dynamics:

Henri Poincare (1854-1912) tried to prove the solar system is stable as a 3-body system. He failed, but won a King Sweden/Norway prize.

He invented the Poincare section. Today, Poincare sections are what we see everyday with a beam position monitor.

He discovered chaos. Today, we study "dynamic apertures".

Collective instabilities:

James Clerk Maxwell (1831-1879) won the Adams Prize when he proved that the Saturn ring is unstable unless it consists of many small solid satellites instead of a single solid piece.

Today, we call this mechanism "negative mass instability".

Gravitational instability is well-known in classical astrophysics, but re-analyzed using modern accelerator physics tools.

If successful, more accelerator physics tools may become available for new insights.

For simplicity,

- Consider nonrelativistic Newtonian gravity. Ignore special relativity is ignored
- The galaxy is in a flat Euclidean space. Also ignore general relativity
- Consider a non-rotating galaxy

Equation of motion

Consider a distribution of stars in a galaxy described by a density $\rho(\vec{x}, \vec{v}, t)$ in phase space $(\vec{x}, \vec{v})$.

Consider one particular star. Equations of motion:

Gravitational constant

Newtonian gravity

$$
\begin{aligned}
\dot{\vec{x}} &= \vec{v} \\
\dot{\vec{v}} &= G \int d\vec{v}' \int d\vec{x}' \, \frac{\rho(\vec{x}', \vec{v}', t)(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^3}
\end{aligned}
$$

The force is obtained by integrating the R^{-2} gravitational forces from all other stars of the galaxy.

These equations do not depend on the mass of the star --- a star or a dust particle are treated the same.

Vlasov equation

Evolution of ρ is described by the Vlasov equation:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \vec{x}} \cdot \dot{\vec{x}} + \frac{\partial \rho}{\partial \vec{v}} \cdot \dot{\vec{v}} \\
=\ &\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \vec{x}} \cdot \vec{v} + \frac{\partial \rho}{\partial \vec{v}} \cdot G \int d\vec{v}' \int d\vec{x}' \frac{\rho(\vec{x}', \vec{v}', t)(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^3} \\
=\ &0
\end{aligned}$$

The force term depends on ρ, making Vlasov equation a nonlinear partial differential integral equation. Our job is to solve it.

$\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3}$ is the "kernel", or the "Green's function". It replaces a "wake function" in accelerator physics.

Perturbation calculation

- Let the galaxy distribution be given by an unperturbed distribution ρ_0 plus some small perturbation $\Delta\rho$.

- Let the unperturbed distribution $\rho_0 = \rho_0(\vec{v})$ be independent of $\vec{x}$, i.e the galaxy has a nominal <u>uniform</u> spatial distribution in $\vec{x}$ extending to infinity.
- The existence of instability is easy to see. Consider a uniform galaxy with all stars stationery but initially having an accidental small local concentration of stars. This extra concentration will grow indefinitely just by the extra gravitational pull.

- Let the perturbation have a structure in t and $\vec{x}$,

$$\rho(\vec{x}, \vec{v}, t) = \rho_0(\vec{v}) + \Delta\rho(\vec{v})\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

ω = frequency in time
k = wave number in space

Dispersion relation

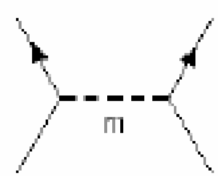
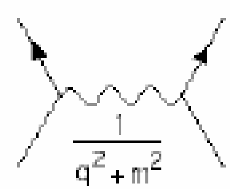
Substitute into Vlasov equation, keeping first order in Δρ (linearization),

$$-i(\omega - \vec{v}\cdot\vec{k})\Delta\rho(\vec{v}) + G\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k}) = 0$$

where $$\vec{q}(\vec{k}) \equiv \int d\vec{x}'\frac{e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}(\vec{x}'-\vec{x})}{|\vec{x}'-\vec{x}|^3}$$

is the Fourier transform of the Newton kernel = “gravitational propagator”,

$$\vec{q}(\vec{k}) = \frac{4\pi i}{|\vec{k}|^2}\vec{k}$$

	kernel in coordinate space	propagator in momentum space
gravitational instability	$\frac{\vec{x}}{\lvert\vec{x}\rvert^3}$	$\frac{4\pi i\vec{k}}{\lvert\vec{k}\rvert^2}$
accelerator physics	wakefields $W_{\parallel}(z), W_{\perp}(z)$	impedances $Z_{\parallel}(\omega), Z_{\perp}(\omega)$
quantum field theory	exchange gauge particles 	propagators 

Rewrite:

$$\Delta\rho(\vec{v}) = -iG\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})}{\omega-\vec{v}\cdot\vec{k}}$$

Integrating both sides over $\vec{v}$ and canceling out $\int d\vec{v}'\,\Delta\rho(\vec{v}')$ gives a dispersion relation

$$1 = -iG\int d\vec{v}\frac{\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})}{\omega-\vec{v}\cdot\vec{k}}$$

Given $\rho_0(\vec{v})$, the dispersion relation can be solved to give ω as function of $\vec{k}$.

=> $\omega(\vec{k})$

Singularity, beware!

Instability

The galaxy is unstable if ω contains a positive imaginary part α. Perturbation $\sim e^{-i\omega t} \sim e^{\alpha t}$ grows with time. An infinitesimal perturbation grows exponentially.

Using the dispersion relation, we obtain $\omega(\vec{k})$ and answer the question:

For what values of $\vec{k}$ is the galaxy unstable?

As will be shown later, the galaxy becomes unstable when $|\vec{k}|$ is less than a threshold value k_{th}. This means galaxies larger than ~ $2\pi/k_{th}$ are unstable. <u>All stable galaxies must be smaller than $2\pi/k_{th}$</u>.

Uniform isotropic galaxy

- Consider a galaxy with isotropic unperturbed distribution:

$$\rho_0 = \rho_0(|\vec{v}|^2)$$

- Normalization:

$$\int_0^\infty 4\pi v^2 dv\, \rho_0(v^2) \;=\; \rho_m$$

in units of g/cm^3

- Dispersion relation becomes

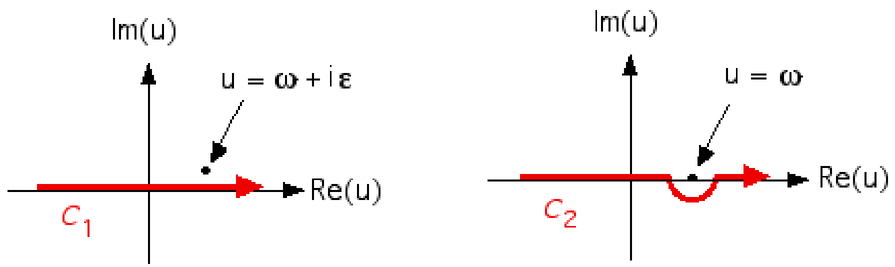
$$1 = \frac{16\pi^2 G}{k} \int_0^\infty v^3 dv\, \rho_0'(v^2) \int_{-1}^{1} du\, \frac{u}{\omega - kvu}$$

Singularity is undefined as is.
Must apply Landau damping analysis, i.e.
$\omega \rightarrow \omega + i\varepsilon$, as in accelerator/plasma physics.

Landau Damping

Analysis amounts to ω –> ω + iε,

$$\int du \frac{1}{u-\omega} \to \int du \frac{1}{u-\omega-i\epsilon} \quad \to \text{P.V.}\left(\int du \frac{1}{u-\omega}\right) - i\pi \times (\text{residue})$$



where P.V. means taking principal value of the integral => singularity is then well-defined.

The seemingly real integral now acquires an explicit imaginary part!

We now specialize the problem further to <u>uniform-sphere isotropic</u> galaxy,

"waterbag model" in accelerator physics

$$\rho_0(v^2) = \begin{cases} \frac{3\rho_m}{4\pi v_0^3} & \text{if } v^2 < v_0^2 \\ 0 & \text{otherwise} \end{cases}$$

v_0^2 is related to the internal "temperature" of the galaxy.

$\Rightarrow$ Dispersion relation becomes

$$\lambda = \frac{1}{2 + x \ln\left|\frac{x-1}{x+1}\right| + i\pi x\, H(1 - |x|)} \qquad (1)$$

where

$$\lambda = \frac{6\pi G \rho_m}{k^2 v_0^2} \quad \text{and} \quad x = \frac{\omega}{k v_0}$$

Problem is solved in principle:

x as function of λ => ω as function of k.

But this requires inverting $\lambda(x)$ to obtain $x(\lambda)$, not easy to do.

Trick is to examine the stability diagram.

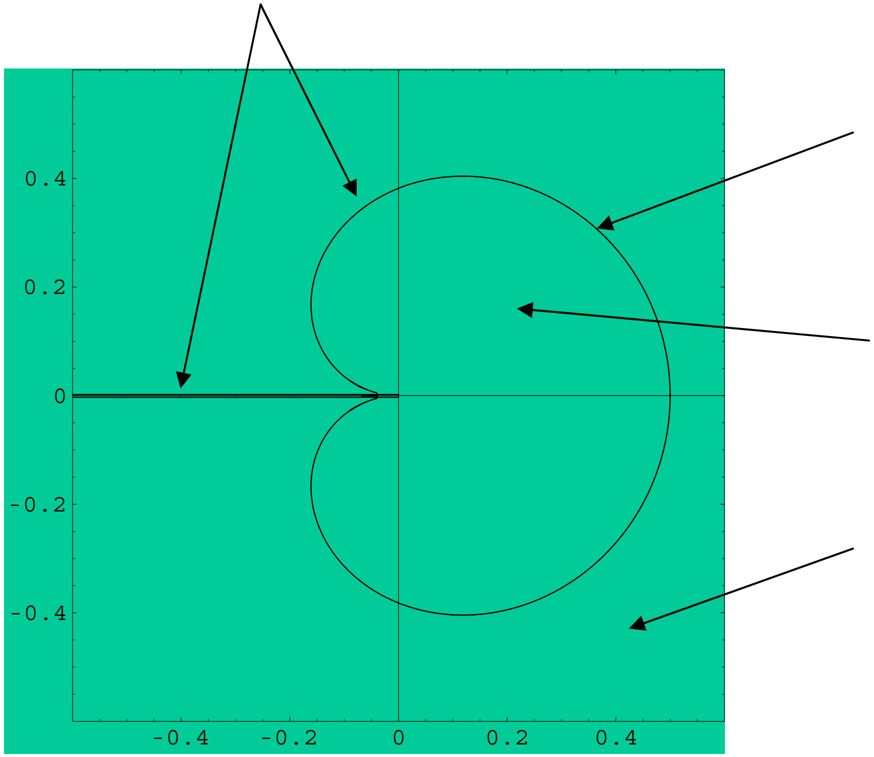
Another standard technique in accelerator physics

Stability diagram

In general, solution x is complex. Instability occurs when x has a positive imaginary part.

At the <u>edge</u> of instability, however, x is (barely) real.

If we scan x = real from $-\infty$ to ∞, the RHS of (1) traces out a curve on a complex plane:



- If λ lies exactly on the trace, the galaxy is at edge of instability.
- If λ lies inside the trace, galaxy is stable.
- If λ lies outside the trace, galaxy is unstable.

In accelerator physics, the LHS of (1) replaces λ by the impedance, which is a complex quantity. The impedance will have to stay inside the stability region if the beam is to be stable. Stability mechanism is Landau damping.

Here, λ is real and positive. The stability condition reduces to

$$\lambda < \frac{1}{2}, \qquad \text{i.e.} \qquad \frac{6\pi G \rho_m}{k^2 v_0^2} < \frac{1}{2}$$

For stability, a galaxy wants to be dilute (small ρ_m) and hot (large v_0). Galaxy size is proportional to its (temperature)$^{1/2}$

Galaxy is stable only if perturbations have

$$k > k_{\text{th}}, \qquad \text{where} \qquad k_{\text{th}} = \frac{\sqrt{12\pi G \rho_m}}{v_0}$$

Stable galaxy must have a size $\lesssim \frac{2\pi}{k_{\text{th}}}$.

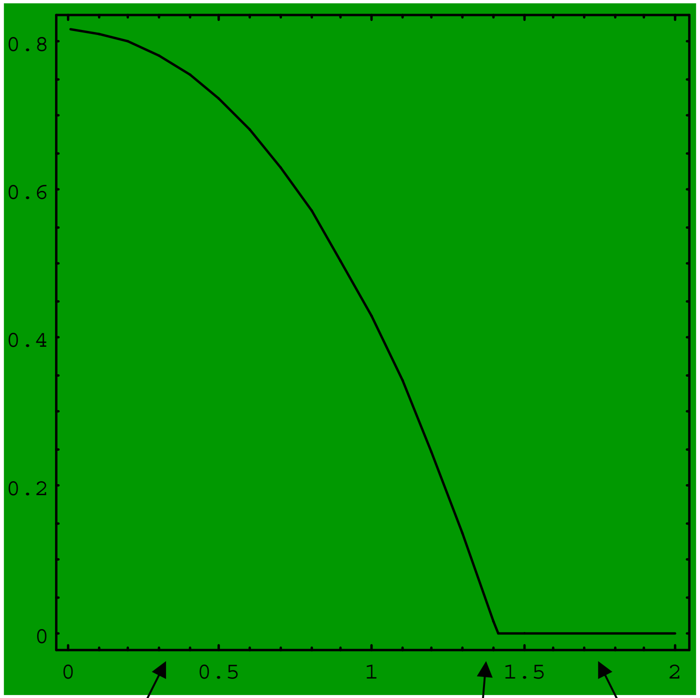
Instability growth rate

How fast does the instability grow when <u>λ > 1/2</u> ?

Growth rate $\tau^{-1} = \mathrm{Im}(\omega)$

$$\frac{6\pi G\rho_m}{k^2 v_0^2} = \frac{1}{2 - \frac{2\tau^{-1}}{kv_0}\tan^{-1}\left(\frac{kv_0}{\tau^{-1}}\right)}$$

$$\frac{\tau^{-1}}{\sqrt{6\pi G\rho_m}}$$



$$\frac{1}{\sqrt{\lambda}} = \frac{kv_0}{\sqrt{6\pi G\rho_m}}$$

λ> 1/2

stability limit
λ = 1/2

λ< 1/2

- Growth is faster for small *k*, i.e. long wavelength.
- Fastest growth when *k* = 0, with

$$(\tau^{-1})_{\max} = \sqrt{4\pi G\rho_m}$$

Numerical example

For the Milky Way, take

$\rho_m = 2 \times 10^{-23}$ g/cm^3
$v_0 = 200$ km/s

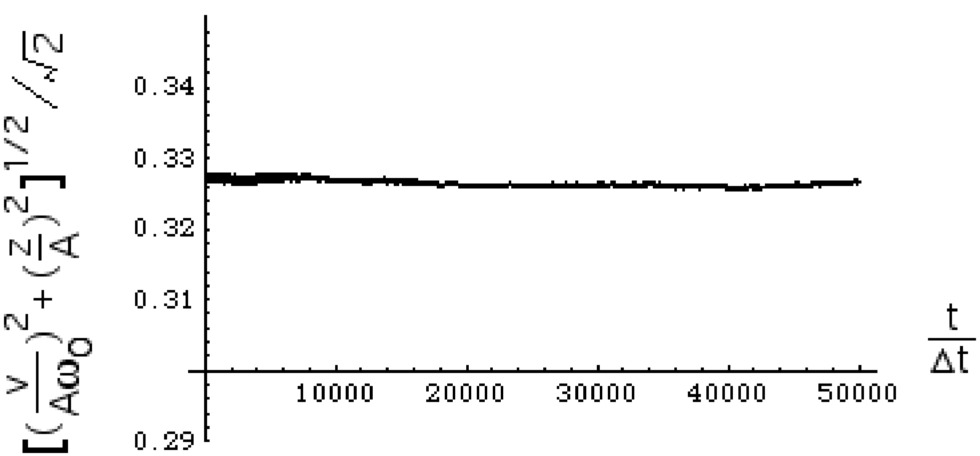
=>

maximum instability growth time $\tau_{max} = 7 \times 10^6$ yrs
stable size = 14,000 light yrs

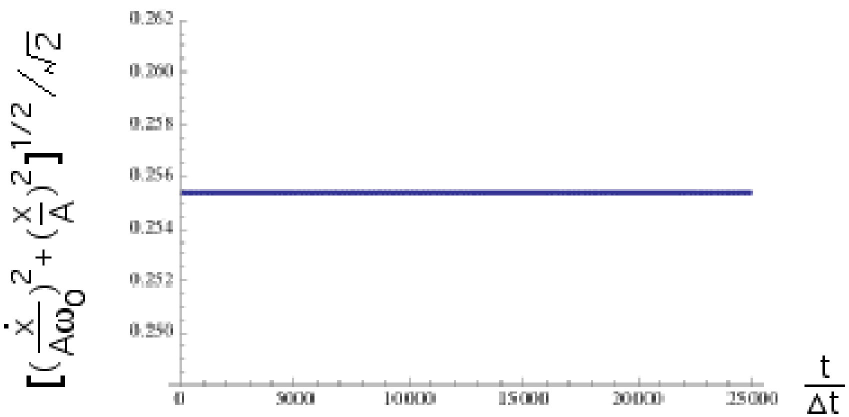
Reasonable agreement with observed size of the Milky Way.

Extensions

- Extend application from (stars in a galaxy) to (galaxies in a galaxy-cluster): Take $\rho_m = 10^{-28}$ g/cm^3, $v_0 = 1000$ km/s, we obtain $\tau_{max} = 3 \times 10^9$ yrs and maximum cluster size = 3×10^7 light-yrs.
- Extension to finite galaxies by a 1-D nonrotating model: Each star is an infinite plane and moves only in the *z*-dimension. Unperturbed distribution has a finite temperature to balance out the gravitational pull. The needed temperature provides stability by Landau damping. Simulation result:



- Extension to a 2-D rotating model: Each star is a line mass infinitely long in *z*-dimension and free to move in the *x*- and *y*-dimensions. Unperturbed distribution is when the rotating centrifugal force exactly balances the gravitational pull. The rotation temperature again stabilizes the galaxy. Simulation result:



- Extension to two colliding galaxies, analogy to the two stream instabilities in accelerator physics.

- Growth for very large wavelengths will be reduced when special relativity is included and action-at-a-distance is avoided.

Summary

- Galaxy instability problem reanalyzed using standard accelerator physics techniques --- Vlasov equation, linearization, dispersion relation, Landau damping, and stability diagrams.

- Dilute, hot galaxies are more stable.

- Isotropic nonrotating stable galaxies must have sizes $< \dfrac{2\pi v_0}{\sqrt{12\pi G\rho_m}}$

- When unstable, the maximum growth rate for long wavelengths

$$(\tau^{-1})_{\max} = \sqrt{4\pi G\rho_m}$$

- Model is simplistic. Extensions are needed/possible.