Stationary distributions of non Gaussian Ornstein–Uhlenbeck processes for beam halos

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1 Introduction

Popularity of **Lévy processes**

(Paul and Baschnagel 1999, Mantegna and Stanley 2001, Barndorff–Nielsen *et al* 2001 and Cont and Tankov 2004):

from **statistical physics**: stable processes

(Bouchaud and Georges 1990, Metzler and Klafter 2000, Paul and Baschnagel 1999, Woyczyński 2001),

to **mathematical finance**: also non stable, *id* processes

(Cont and Tankov 2004 and references quoted therein).

stable processes: selfsimilarity, but if non gaussian

- infinite variance (*truncated* distributions);
- the x decay rates of the *pdf*'s can not exceed x^{-3}.

infinitely divisible **(*id*) processes**: stationarity vs. selfsimilarity, the *pdf* of increments could be known only at one time scale, but new applications in the physical domain begin to emerge (Cufaro Petroni *et al* 2005, 2006, Vivoli *et al* 2006):
the motion in the charged particle **accelerator beams** points to a ***id*, Student, Ornstein–Uhlenbeck (*OU*) process**.

Dynamical description: **stochastic mechanics (*sm*)** (Nelson 1967, 1985, Guerra 1981, Guerra and Morato 1983) suitable for many *controlled, time–reversal invariant* systems (Albeverio, Blanchard and Høgh-Krohn 1983, Paul and Baschnagel 1999, Cufaro Petroni *et al* 1999, 2000, 2003, 2004)
Generalize *sm* to the non Gaussian Lévy noises:
a ***sm* with jumps** to produce **halos** in accelerator beams.

Lévy processes already have applications in **quantum** domain:

- spinning particles (De Angelis and Jona–Lasinio 1982)
- relativistic quantum mechanics (De Angelis 1990)
- stochastic quantization (Albeverio, Rüdiger and Wu 2001).

Here: not only quantum systems, but also *general complex systems (particle beams) with dynamical control.*
Only *one dimensional models*, without going into the problem of the dependence structure of a multivariate process

At present no Lévy *sm* is available: we just have *OU* processes
Possible underlying Lévy noises: **non stable, selfdecomposable, Student processes**
Student laws $\mathcal{T}(\nu, \delta)$ are not closed under convolution:
the noise distribution will not be Student at every time t.

Results for a $\nu = 3$ Student noise (Cufaro Petroni 2007a, 2007b):

1. for integer times $t = n$ **the noise transition law is a mixture of a finite number of Student laws**; only at $t = 1$ this law is exactly $\mathcal{T}(3, \delta)$;

2. for every finite time t **the noise *pdf* asymptotic behavior always is the same (x^{-4}) as that of the $\mathcal{T}(3, \delta)$ law**; this is the behavior put in evidence by Vivoli *et al* 2006 in the solutions of the complex dynamical system used to study the **beams** of charged particles in accelerators.

3. **the stationary distribution** of the OU process with $\mathcal{T}(3, \delta)$ noise can be calculated, and **its asymptotic behavior again is x^{-4}**.

2 Lévy processes generated by *id* laws

Lévy process $X(t)$: a *stationary, stochastically continuous, independent increment* Markov process.

Given a ***type* of centered, *id* distributions** with *chf*'s $\varphi(au)$ $(a > 0)$, the ***chf* of the transition law** of in the time interval $[s, t]$ $(T > 0)$ is

$$\Phi(au, t-s) = [\varphi(au)]^{(t-s)/T} \tag{1}$$

and the **transition *pdf*** with initial condition $X(s) = y$, $\mathbb{P}$-q.o.

$$p(x, t|\, y, s) = \frac{1}{2\pi} \lim_{M\to+\infty} \int_{-M}^{M} [\varphi(au)]^{(t-s)/T}\, e^{-i(x-y)u}\, du \tag{2}$$

Along the evolution **stable laws** remain within the **same type**
and the process is **selfsimilar**

However, all the *non gaussian stable laws*:

- do not have a finite variance;
- show a rather restricted range of possible decays for large x.

Non stable, *id* laws have none of these shortcomings
but the Lévy processes show *no selfsimilarity*.

When **closed under convolution**:
the evolution is in the time dependence of some parameter,
but the laws do not belong to the *same type*.

When **not even closed under convolution**:
the transition laws do not remain within the same family
the evolution is not just in the time dependence of parameters.

Student/Variance Gamma laws are *id*, non stable

Student laws are not even closed under convolution, but

- they are *selfdecomposable*
- they can have a *wide range of decay laws* for $|x| \to +\infty$;
- they can have *finite variance*

Selfdecomposable laws have two relevant properties:

1. can produce *non stationary, selfsimilar, additive processes*
2. alternatively can produce *Lévy processes*
3. are the *limit laws of Ornstein–Uhlenbeck processes*

When $\sigma^2 < +\infty$, ***id* Lévy processes** have variance $\sigma^2 t/T$:
ordinary (non anomalous) diffusions

3 Particular classes of *id* distributions

Variance Gamma (VG) laws $\mathcal{VG}(\lambda, \alpha)$ ($\lambda > 0$ and $\alpha > 0$):

$$f_{VG}(x) = \frac{2\alpha}{2^\lambda \Gamma(\lambda)\sqrt{2\pi}} (\alpha|x|)^{\lambda - \frac{1}{2}} K_{\lambda - \frac{1}{2}}(\alpha|x|)$$

$$\varphi_{VG}(u) = \left(\frac{\alpha^2}{\alpha^2 + u^2}\right)^\lambda$$

α is a spatial scale parameter; λ classifies different types.

$\mathcal{VG}(1, \alpha)$ are the Laplace (double exponential) laws $\mathcal{L}(\alpha)$

$$f(x) = \frac{\alpha}{2} e^{-\alpha|x|}, \qquad \varphi(u) = \frac{\alpha^2}{\alpha^2 + u^2}$$

Asymptotic behavior of $\mathcal{VG}(\lambda, \alpha)$ is $(\alpha|x|)^{\lambda-1} e^{-\alpha|x|}$

The VG laws are ***id*, selfdecomposable, but not stable**.

The $\mathcal{VG}(\lambda, \alpha)$ with a fixed α are **closed under convolution**:

$$\mathcal{VG}(\lambda_1, \alpha) \star \mathcal{VG}(\lambda_2, \alpha) = \mathcal{VG}(\lambda_1 + \lambda_2\,, \alpha)$$

Student laws $\mathcal{T}(\nu, \delta)$ ($\nu > 0$, $\delta > 0$ and $B(z, w)$ Beta function)

$$f_{ST}(x) = \frac{1}{\delta\, B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \left(\frac{\delta^2}{\delta^2 + x^2}\right)^{\frac{\nu+1}{2}} \qquad (3)$$

$$\varphi_{ST}(u) = 2\, \frac{(\delta|u|)^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(\delta|u|)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} \qquad (4)$$

δ is a spatial scale parameter; ν classifies different types.

Asymptotic behavior of $\mathcal{T}(\nu, \delta)$ is $(|x|/\delta)^{-\nu-1}$

$\mathcal{T}(1,\delta)$ class of the Cauchy $\mathcal{C}(\delta)$ laws

$$f(x) = \frac{1}{\delta\,\pi}\,\frac{\delta^2}{\delta^2 + x^2}\,, \qquad\qquad \varphi(u) = e^{-\delta|u|}$$

The Student distributions $\mathcal{T}(\nu,\delta)$ are ***id*, selfdecomposable, but not stable** with one notable exception:
the Cauchy laws $\mathcal{T}(1,\delta) = \mathcal{C}(\delta)$.
The Student laws are **not even closed under convolution**.

4 The VG and Student processes

The **transition *chf*** for a $\mathcal{VG}(\lambda)$ process ($\alpha = 1$, $T = 1$, $s = 0$ and $y = 0$) is:

$$\Phi(u, t|\lambda) = [\varphi_{VG}(u)]^t = \left(\frac{1}{1+u^2}\right)^{\lambda t} \tag{5}$$

Increment law in $[0, t]$ is $X(t) \sim \mathcal{VG}(\lambda t)$ and the ***pdf*** is

$$p(x, t|\lambda) = \frac{2}{2^{\lambda t}\Gamma(\lambda t)\sqrt{2\pi}}\,|x|^{\lambda t - \frac{1}{2}} K_{\lambda t - \frac{1}{2}}(|x|) \tag{6}$$

Asymptotic behavior (all the moments exist)

$$p(x, t|\lambda) \sim |x|^{\lambda t - 1}\, e^{-|x|}, \qquad\qquad |x| \to +\infty$$

The Student family $\mathcal{T}(\nu, \delta)$ is not closed under convolution

An explicit form of the transition *pdf* not available

We study the $\nu = 3$, $\mathcal{T}(3, \delta)$ *Student process* candidate to describe the increments in the velocity process for particles in an accelerator beam (Vivoli *et al* 2006).

For $\delta = 1$, the $\mathcal{T}(3, 1)$–process has transition *pdf*

$$p(x, t|\,3) = \Re\left\{\frac{e^{t+ix}\,\Gamma(t+1, t+ix)}{\pi(t+ix)^{t+1}}\right\} \tag{7}$$

with $\Gamma(a, z)$ the incomplete Gamma function, and

$$p(x, t|\,3) = \frac{2t}{\pi x^4} + o\left(|x|^{-4}\right), \qquad |x| \to +\infty \qquad (t > 0)$$

For fixed, finite $t > 0$ the asymptotic behavior of $p(x, t|\,3)$ is always infinitesimal at the same order $|x|^{-4}$ of the original $\mathcal{T}(3, 1)$

At integral times $t = n = 1, 2, \ldots$ the transition *pdf* $p(x, n|\,3)$ of the $\mathcal{T}(3, 1)$–Student process is a mixture of Student *pdf*'s with

- odd integer orders $\nu = 2k + 1$ with $k = 0, 1, \ldots$,
- integer scaling factors $\delta = n$,
- relative weights

$$q_n(k|\,3) = \frac{(-1)^k}{2k+1} \sum_{j=0}^{2k+1} \binom{n}{j} \binom{2k+1}{j} \binom{j}{k} (j+1)! \left(\frac{-1}{2n}\right)^j$$

namely: mixtures of $\mathcal{T}(2k+1, n)$ laws with $k = 1, \ldots, n$ with no Student law of order smaller than $\nu = 3$

The distributions $q_n(k|\,3)$ are displayed in Figure 1.

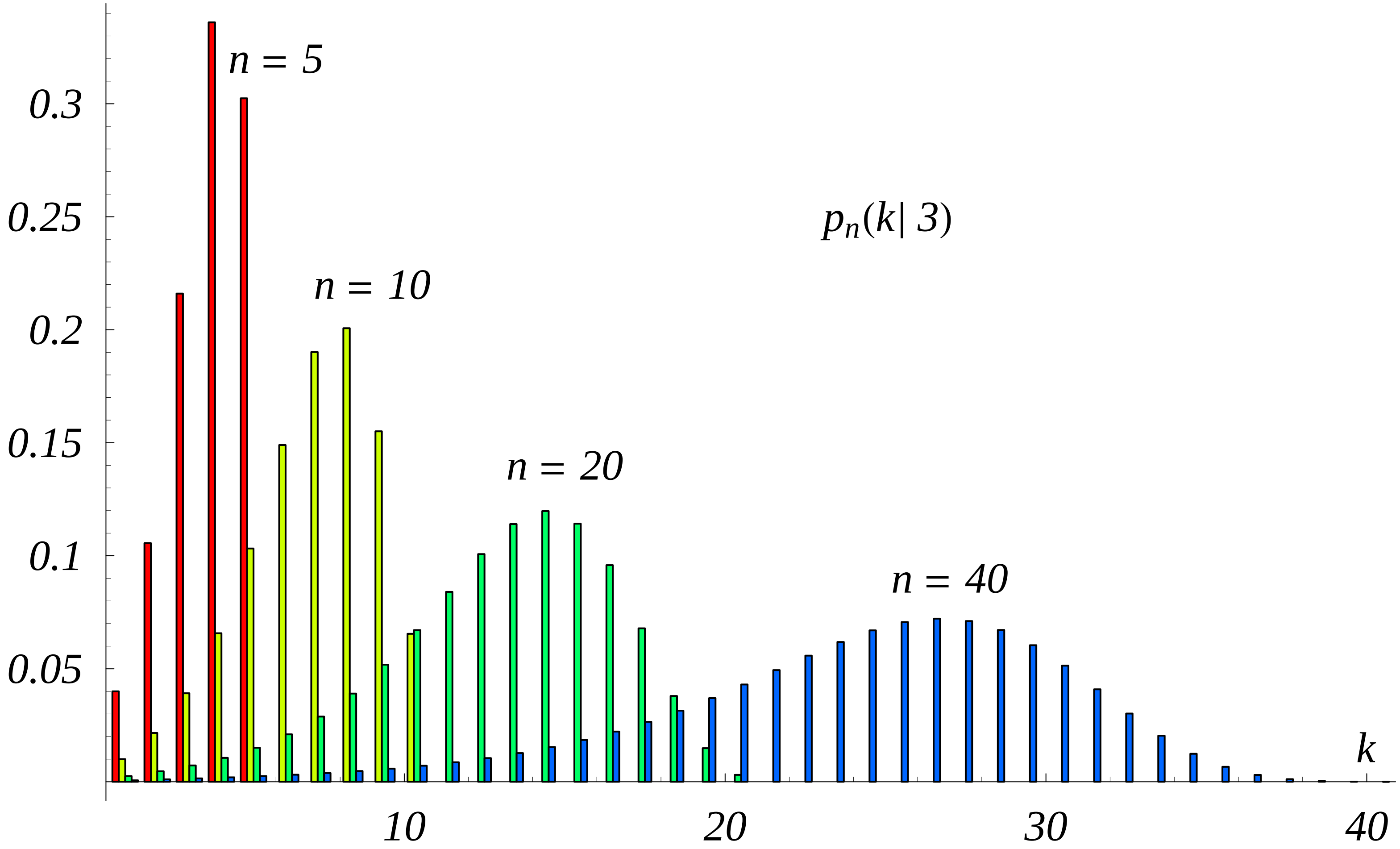

Figure 1: Mixture weights of the integer time $(t = n)$ components for a Student process with $\nu = 3$.

5 Ornstein–Uhlenbeck processes

The VG and Student processes have no Brownian component: they are **pure jump processes**

The VG and Student processes have *infinite activity*: namely the set of jump times is countably infinite and dense in $[0, +\infty]$.

At first sight the **simulated samples** of both a VG and a Student process do not look very different from that of a Wiener process.

Lévy diffusions $Y(t)$:

solutions of *SDE* driven by a Lévy process $X(t)$

$$dY(t) = \alpha(t, Y(t))\, dt + dX(t)$$

Compare **Ornstein–Uhlenbeck (OU) processes** driven by either VG or $\mathcal{T}(3,\delta)$ noises $X(t)$

$$dY(t) = -b\,Y(t)\,dt + dX(t) \tag{8}$$

with usual OU process driven by Brownian noise $B(t)$

$$dY(t) = -b\,Y(t)\,dt + dB(t) \tag{9}$$

Figure 2: samples of 5 000 steps with noise laws in the Table

(a)	(b)	(c)
$\mathcal{N}(0,1)$	$\mathcal{VG}(1,\sqrt{2})$	$\mathcal{T}(3,1)$
$\frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}$	$\frac{1}{\sqrt{2}}\,e^{-\sqrt{2}\,\lvert x\rvert}$	$\frac{2}{\pi}\,\frac{1}{(1+x^2)^2}$

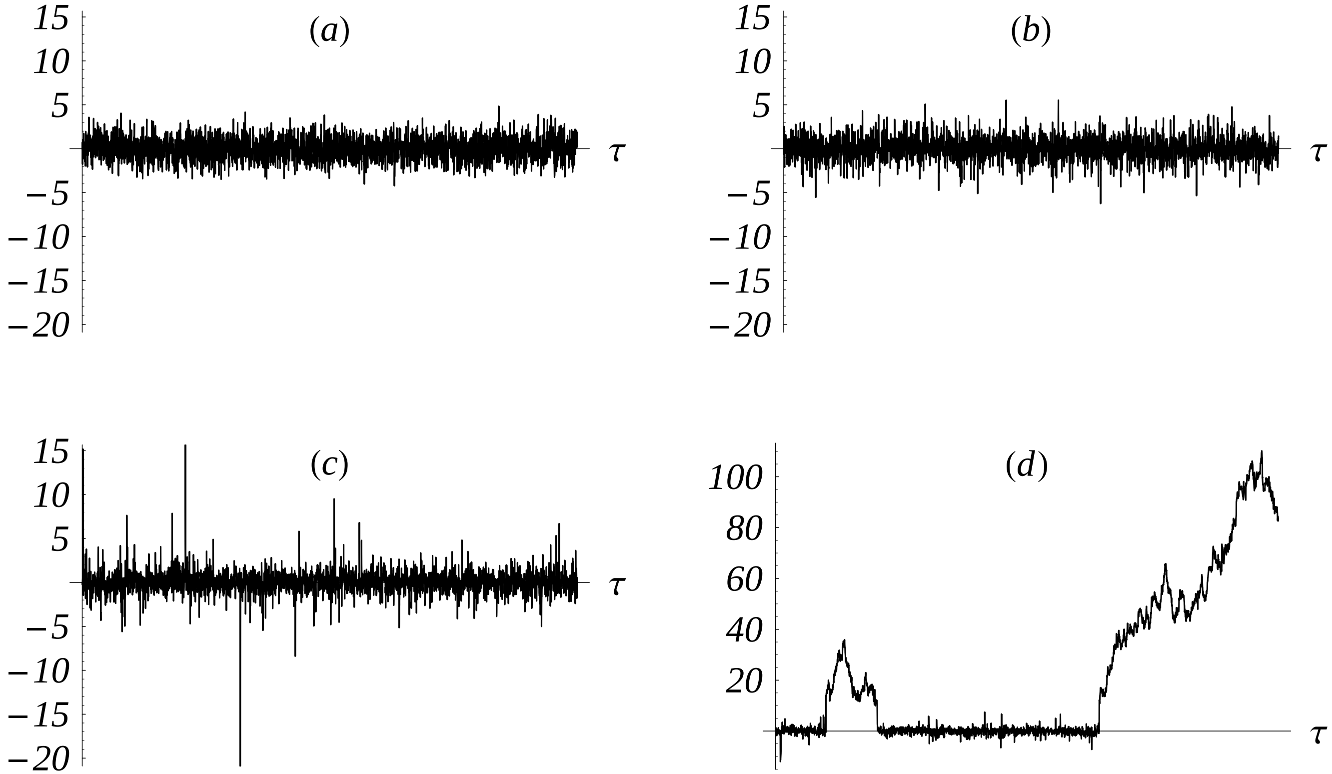

Figure 2: OU processes driven by (a) Brownian motion; (b) VG Lévy noise; (c) Student Lévy noise; and then (d) Student OU–type process with restoring force of finite range.

(a) typical OU process driven by normal Brownian motion,
(b) and (c) OU–type processes driven by VG and Student noises

(a) is rather strictly confined by the restoring force $-by$
(b) and (c) show **spikes** going outside the confining region.

(d) take a *restoring force of a finite range*:

$$\begin{aligned} dY(t) &= \alpha(Y(t))\,dt + dX(t) \\ \alpha(y) &= \begin{cases} -by, & \text{for } |y| \leq q; \\ 0, & \text{for } |y| > q. \end{cases} \qquad q > 0 \end{aligned}$$

The restoring force acts only in $[-q, q]$
When the process jumps beyond $y = \pm q$ it diffuses freely:
possible model of *halo formation in particle beams*

Role of **selfdecomposability in OU processes**:

if $Y(t)$ is solution of the OU SDE

$$dY(t) = -bY(t)\,dt + dX(t)$$

for a Lévy noise $X(t)$ with logarithmic characteristic $\psi(u) = \log\varphi(u)$, then the stationary distribution is absolutely continuous and selfdecomposable with logarithmic characteristic $\psi_Y(u)$ such that

$$\psi_Y(u) = \int_0^\infty \psi(ue^{-bt})\,dt\,, \qquad \psi(u) = bu\psi_Y'(u)$$

VG and Student laws are *id* and selfdecomposable

Then we can explicitly find the stationary laws of the OU processes with VG and Student noises.

OU stationary distribution for a Laplace law $\mathcal{VG}(1, \sqrt{2}/a)$

with variance $\sigma_X^2 = a^2$:

$$\psi(u) = -\log\left(1 + \frac{a^2u^2}{2}\right), \qquad \psi_Y(u) = \frac{1}{2b}\,\mathrm{Li}_2\left(-\frac{a^2u^2}{2}\right)$$

where *dilogarithm* is

$$\mathrm{Li}_2(x) = \int_x^0 \frac{\log(1-s)}{s}\,ds \qquad \left(= \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \qquad |x| \le 1\right)$$

variance of the stationary distribution

$$\sigma_Y^2 = -\varphi_Y''(0) = \frac{a^2}{2b}$$

pdf can be numerically evaluated

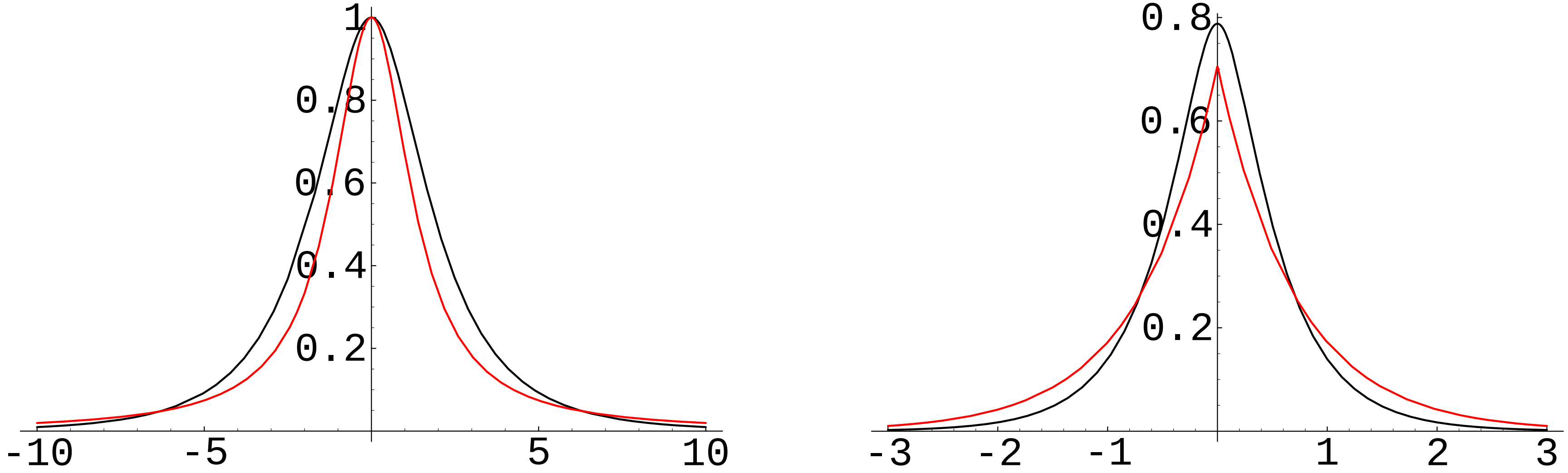

Figure 3: *chf* and *pdf* of a OU stationary distribution (black lines), compared with the *chf* and *pdf* of the driving $\mathcal{VG}(1, \sqrt{2}/a)$ Laplace noise (red lines). Here $a = b = 1$.

OU stationary distribution for a student law $\mathcal{T}(3, a)$
with variance $\sigma_X^2 = a^2$:

$$\psi(u) = -a|u| + \log(1 + a|u|)\,, \qquad \psi_Y(u) = -\frac{a|u|}{b} - \frac{1}{b}\,\mathrm{Li}_2(-a|u|)$$

variance of the stationary distribution

$$\sigma_Y^2 = -\varphi_Y''(0) = \frac{a^2}{2b}$$

pdf $f(x)$ can be numerically evaluated and

$$f(x) \sim 0.4244 \times x^{-4}\,, \qquad x \to \infty$$

namely: *the stationary solution is not a Student law, but it keeps the same asymptotic behavior of the driving Student noise.*

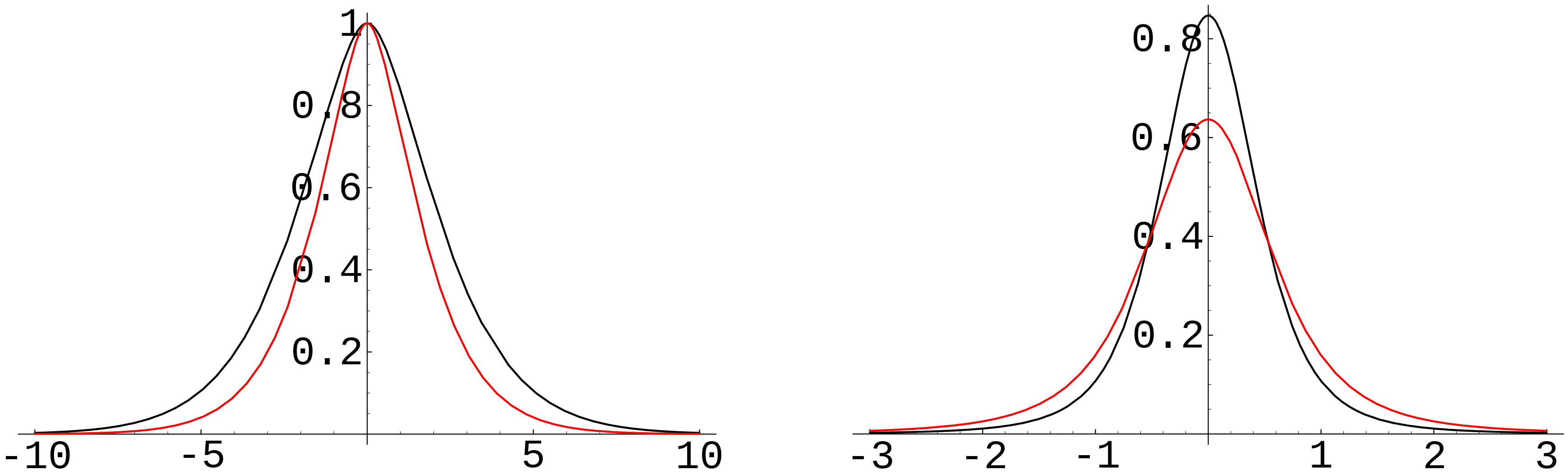

Figure 4: *chf* and *pdf* of a OU stationary distribution (black lines), compared with the *chf* and *pdf* of the driving $\mathcal{T}(3, a)$ Student noise (red lines). Here $a = b = 1$.

6 Conclusions

- An OU process driven by a *selfdecomposable* $\mathcal{T}(3, a)$ Student noise seems to be a good candidate as *a model for halo formation in beams of charged particles in accelerators*

- The driving Student noise and the stationary laws show the same *asymptotic behavior* (x^{-4}) of dynamical simulations

- Selfdecomposable processes are at present under intense scrutiny for possible use in *option pricing* (Carr *et al* 2007)

- A *dynamical model (SM) for processes driven by non Gaussian Lévy noises* must now be elaborated in order to achieve a reasonable control of the beam size.

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