SPIN-ORBITAL MOTION: SYMMETRY AND DYNAMICS

A. Fedorova, M. Zeitlin, IPME, RAS, St. Petersburg, V.O. Bolshoj pr., 61, 199178, Russia [*][†]

Abstract

We present the applications of variational–wavelet approach to nonlinear (rational) model for spin-orbital motion: orbital dynamics and Thomas-BMT equations for classical spin vector. We represent the solution of this dynamical system in framework of periodical wavelets via variational approach and multiresolution.

1 INTRODUCTION

In this paper we consider the applications of a new numerical-analytical technique which is based on the methods of local nonlinear harmonic analysis or wavelet analysis to the spin orbital motion. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and give for the general type of operators (differential, integral, pseudodifferential) in such bases the maximum sparse forms. Our approach in this paper is based on the generalization of variational-wavelet approach from [1]-[8], which allows us to consider not only polynomial but rational type of nonlinearities [9]. The solution has the following form

$$z(t) = z_N^{slow}(t) + \sum_{j \geq N} z_j(\omega_j t), \quad \omega_j \sim 2^j \quad (1)$$

which corresponds to the full multiresolution expansion in all time scales. Formula (1) gives us expansion into a slow part z_N^{slow} and fast oscillating parts for arbitrary N. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first term in the RHS of equation (1) corresponds on the global level of function space decomposition to resolution space and the second one to detail space. In this way we give contribution to our full solution from each scale of resolution or each time scale. The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution.

In part 2 we consider spin-orbital motion. In part 3 starting from variational formulation we construct via multiresolution analysis explicit representation for all dynamical variables in the base of compactly supported periodized wavelets. In part 4 we consider results of numerical calculations.

2 SPIN-ORBITAL MOTION

Let us consider the system of equations for orbital motion and Thomas-BMT equation for classical spin vector [10]:

$\mathrm{d}q/\mathrm{d}t = \partial H_{orb}/\partial p$, $\quad \mathrm{d}p/\mathrm{d}t = -\partial H_{orb}/\partial q$, $\quad \mathrm{d}s/\mathrm{d}t = w \times s$, where

$$
\begin{aligned}
H_{orb} &= c\sqrt{\pi^2 + m_0 c^2} + e\Phi, \\
w &= -\frac{e}{m_0 c\gamma}(1 + \gamma G)\vec{B} \\
&\quad + \frac{e}{m_0^3 c^3 \gamma}\frac{G(\vec{\pi} \cdot \vec{B})\vec{\pi}}{(1+\gamma)} \\
&\quad + \frac{e}{m_0^2 c^2 \gamma}\frac{G + \gamma G + 1}{(1+\gamma)}[\pi \times E],
\end{aligned}
\quad (2)
$$

$q = (q_1, q_2, q_3), p = (p_1, p_2, p_3)$ are canonical position and momentum, $s = (s_1, s_2, s_3)$ is the classical spin vector of length $\hbar/2$, $\pi = (\pi_1, \pi_2, \pi_3)$ is kinetic momentum vector. We may introduce in 9-dimensional phase space $z = (q, p, s)$ the Poisson brackets $\{f(z), g(z)\} = f_q g_p - f_p g_q + [f_s \times g_s] \cdot s$ and the Hamiltonian equations are $\mathrm{d}z/\mathrm{d}t = \{z, H\}$ with Hamiltonian

$$H = H_{orb}(q, p, t) + w(q, p, t) \cdot s. \quad (3)$$

More explicitly we have

$$
\begin{aligned}
\frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{\partial H_{orb}}{\partial p} + \frac{\partial(w \cdot s)}{\partial p} \\
\frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{\partial H_{orb}}{\partial q} - \frac{\partial(w \cdot s)}{\partial q} \\
\frac{\mathrm{d}s}{\mathrm{d}t} &= [w \times s]
\end{aligned}
\quad (4)
$$

We will consider this dynamical system in [11] via invariant approach, based on consideration of Lie-Poisson structures on semidirect products. But from the point of view which we used in [9] we may consider the similar approximations and then we also arrive to some type of polynomial/rational dynamics.

3 VARIATIONAL WAVELET APPROACH FOR PERIODIC TRAJECTORIES

We start with extension of our approach to the case of periodic trajectories. The equations of motion corresponding to our problems may be formulated as a particular case of the general system of ordinary differential equations $dx_i/dt = f_i(x_j, t)$, $(i, j = 1, ..., n)$, $0 \leq t \leq 1$, where f_i are not more than rational functions of dynamical variables x_j and have arbitrary dependence of time but with periodic boundary conditions. According to our variational approach we have the solution in the following form

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t), \qquad x_i(0) = x_i(1), \quad (5)$$

where λ_i^k are the roots of reduced algebraical systems of equations with the same degree of nonlinearity and $\varphi_k(t)$ corresponds to useful type of wavelet bases (frames). It should be noted that coefficients of reduced algebraical system are the solutions of additional linear problem and also depend on particular type of wavelet construction and type of bases.

Our constructions are based on multiresolution approach. Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. We consider a r-regular multiresolution analysis of $L^2(\mathbf{R}^n)$ (of course, we may consider any different functional space) which is a sequence of increasing closed subspaces V_j:

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ... \qquad (6)$$

Then just as V_j is spanned by dilation and translations of the scaling function, so W_j are spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$, where

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k). \qquad (7)$$

All expansions, which we used, are based on the following properties:

$$L^2(\mathbf{R}) = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j} \qquad (8)$$

We need also to find in general situation objects

$$\Lambda_{\ell_1 \ell_2 ... \ell_n}^{d_1 d_2 ... d_n} = \int\limits_{-\infty}^{\infty} \prod \varphi_{\ell_i}^{d_i}(x)\mathrm{d}x, \qquad (9)$$

but now in the case of periodic boundary conditions. Now we consider the procedure of their calculations in case of periodic boundary conditions in the base of periodic wavelet functions on the interval [0,1] and corresponding expansion (1) inside our variational approach. Periodization procedure gives us

$$\hat{\varphi}_{j,k}(x) \equiv \sum_{\ell \in Z} \varphi_{j,k}(x - \ell) \qquad (10)$$

$$\hat{\psi}_{j,k}(x) = \sum_{\ell \in Z} \psi_{j,k}(x - \ell)$$

So, $\hat{\varphi}, \hat{\psi}$ are periodic functions on the interval [0,1]. Because $\varphi_{j,k} = \varphi_{j,k'}$ if $k = k'\mathrm{mod}(2^j)$, we may consider only $0 \le k \le 2^j$ and as consequence our multiresolution has the form $\bigcup_{j \ge 0} \hat{V}_j = L^2[0,1]$ with $\hat{V}_j = \mathrm{span}\{\hat{\varphi}_{j,k}\}_{k=0}^{2j-1}$ [12]. Integration by parts and periodicity gives useful relations between objects (9) in particular quadratic case $(d = d_1 + d_2)$:

$$\Lambda_{k_1,k_2}^{d_1,d_2} = (-1)^{d_1}\Lambda_{k_1,k_2}^{0,d_2+d_1}, \qquad (11)$$

$$\Lambda_{k_1,k_2}^{0,d} = \Lambda_{0,k_2-k_1}^{0,d} \equiv \Lambda_{k_2-k_1}^d$$

So, any 2-tuple can be represented by Λ_k^d. Then our second additional linear problem is reduced to the eigenvalue problem for $\{\Lambda_k^d\}_{0 \le k \le 2^j}$ by creating a system of 2^j homogeneous relations in Λ_k^d and inhomogeneous equations. So, if we have dilation equation in the form $\varphi(x) = \sqrt{2}\sum_{k \in Z} h_k \varphi(2x - k)$, then we have the following homogeneous relations

$$\Lambda_k^d = 2^d \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} h_m h_\ell \Lambda_{\ell+2k-m}^d, \qquad (12)$$

or in such form $A\lambda^d = 2^d\lambda^d$, where $\lambda^d = \{\Lambda_k^d\}_{0 \le k \le 2^j}$. Inhomogeneous equations are:

$$\sum_\ell M_\ell^d \Lambda_\ell^d = d!2^{-j/2}, \qquad (13)$$

where objects $M_\ell^d(|\ell| \le N - 2)$ can be computed by recursive procedure

$$M_\ell^d = 2^{-j(2d+1)/2}\tilde{M}_\ell^d, \qquad (14)$$

$$\tilde{M}_\ell^k = <x^k, \varphi_{0,\ell}> = \sum_{j=0}^k \binom{k}{j} n^{k-j} M_0^j, \quad \tilde{M}_0^\ell = 1.$$

So, we reduced our last problem to standard linear algebraical problem. Then we use the methods from [9]. As a result we obtained for closed trajectories of orbital dynamics the explicit time solution (1) in the base of periodized wavelets (10).
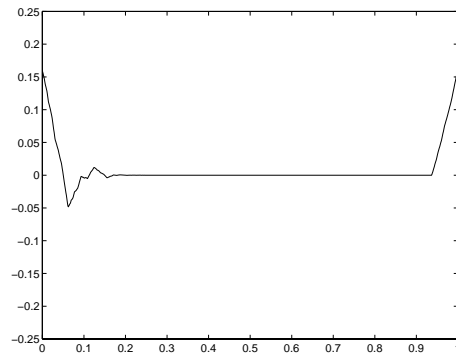

Figure 1: Periodic wavelet

4 NUMERICAL CALCULATIONS

In this part we consider numerical illustrations of previous analytical approach. Our numerical calculations are based on periodic compactly supported Daubechies wavelets and related wavelet families (Fig. 1). Also in our modelling we added noise as perturbation to our spin orbit configurations.

On Fig. 2 we present according to formulae (2),(6) contributions to approximation of our dynamical evolution (top row on the Fig. 3) starting from the coarse approximation, corresponding to scale 2^0 (bottom row) to the finest one corresponding to the scales from 2^1 to 2^5 or from slow

to fast components (5 frequencies) as details for approximation. Then on Fig. 3, from bottom to top, we demonstrate the summation of contributions from corresponding levels of resolution given on Fig. 2 and as result we restore via 5 scales (frequencies) approximation our dynamical process(top row on Fig. 3). The same decomposition/approximation we produce also on the level of power spectral density in the process with noise (Fig. 4).
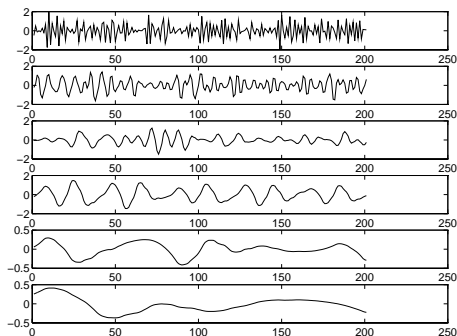

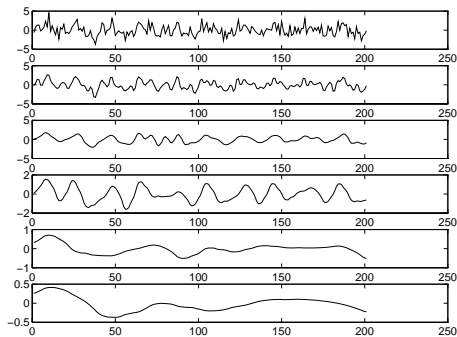Figure 2: Contributions to approximation: from scale 2^1 to 2^5 (with noise).

Figure 3: Approximations: from scale 2^1 to 2^5 (with noise).
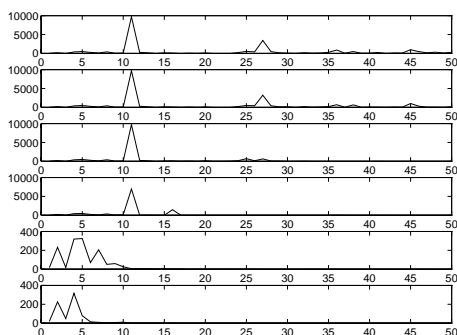

Figure 4: Power spectral density: from scale 2^1 to 2^5 (with noise)

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