Combinations of Skew-Quadrupoles for Local Coupling Control

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Abstract

A recipe is proposed to make a knob of a group of skew-quadrupoles for local tuning of the $x-y$ coupling parameters which should leave the outer lattice uncoupled. Analysis made to first order in the skew-quadrupole strength shows that a string of five kicks is needed in general, while in some special cases with certain phase relations between the locations of the kicks, a lesser number of correctors will do the job. In the used approximation, the skew-quadrupole knobs act in superposition.

1 INTRODUCTION

This paper gives derivation of a rule to compose a local-coupling control combination of skew-quadrupoles with appropriate relations of the skew-gradient between them, given their locations in the linear lattice. The requirement is that when applied to the lattice without $x-y$ coupling, the outer lattice functions should remain intact. This means that the optics string with skew-quadrupoles aimed at local control of coupling should be transparent, i.e. its 4×4 transport matrix should be identical to the nominal one. We find this rule in the first order of the skew-quadrupole strength, assuming the lengths of the skew-quadrupoles short enough to use the kick approximation. These approximations are relevant to development of the weak coupling correction procedures, which are important for the operation of e^+e^- colliders with flat beams.

2 LINEAR COUPLING ANALYSIS

2.1 Equations of Motion

Starting from the conventional equations of the paraxial motion (with the trajectory slopes $x', y' \ll 1$) in the right-handed tripod (x, s, y), where x and y are the horizontal and vertical deviations of the trajectory from the design orbit having the local curvature radius of ρ, and s is the path along the design orbit, we write:

$$\begin{aligned} x'' + K_x x &= -(q + L'/2)\, y - L\, y'\,, \\ y'' + K_y y &= -(q - L'/2)\, x + L\, x'\,. \end{aligned} \qquad (1)$$

Here the prime sign denotes differentiation with respect to s, and the guide magnetic fields (B_x, B_s, B_y) in the Lorentz force are linearized in small neighborhood of the design orbit, to give the focusing functions,

$$K_x = \frac{1}{\rho^2} - \frac{e}{pc}\frac{\partial B_y}{\partial x}\,, \qquad K_y = \frac{e}{pc}\frac{\partial B_y}{\partial x}\,, \qquad (2)$$

and the coupling coefficients L, L' and q representing longitudinal field in solenoids, solenoid end-field effect and skew-quadrupoles, respectively:

$$\begin{aligned} L &= \frac{eB_s}{pc}\,, \qquad L' = -\frac{e}{pc}\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right), \\ q &= \frac{e}{2pc}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right). \end{aligned} \qquad (3)$$

Here p is the particle momentum; $pc + eB_0\rho = 0$. Use has been made of Maxwell's equations in free space

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0; \qquad \frac{\partial B_s}{\partial s} = -\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right).$$

2.2 Equations for the Betatron Amplitudes

The general lattice analysis tools fully accounting for linear coupling are available in literature, offering different generalizations of the Courant-Snyder approach. Here we restrict ourselves to the complex amplitude technique, where the lattice functions of ideal uncoupled lattice give a basis for description of a coupled lattice.

In an AG lattice analysis we are going to employ the Floquet function formalism [1]. We introduce special complex solutions $f_x,\ f_y$ to the uncoupled equations, called the Floquet functions:

$$f''_{x,y} + K_{x,y} f_{x,y} = 0\,, \qquad (4)$$

Their moduli are periodic with the period of the focusing functions $K_{x,y}$, and the Wronskians are normalized to $-2i$:

$$f_{x,y} f^{*\prime}_{x,y} - f'_{x,y} f^*_{x,y} = \begin{vmatrix} f_{x,y} & f^*_{x,y} \\ f'_{x,y} & f^{*\prime}_{x,y} \end{vmatrix} = -2i\,. \qquad (5)$$

The Floquet functions may be expressed in terms of the Twiss parameters β, α and ψ of the uncoupled lattice:

$$\begin{pmatrix} f_{x,y} \\ f'_{x,y} \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_{x,y}} \\ (i - \alpha_{x,y})/\sqrt{\beta_{x,y}} \end{pmatrix} \exp(i\psi_{x,y}). \qquad (6)$$

Now we turn back to the coupled equations (1), and substitute for their solutions:

$$x = A_x f_x + \text{c.c.}, \quad y = A_y f_y + \text{c.c.}, \qquad (7)$$

having in mind that due to the coupling on the right hand side of (1), the complex amplitudes $A_{x,y}$ are now variables rather than constants.

Due to redundancy in the substitution of a real function x with two complex functions $A_x,\ f_x$, we are free to choose an additional relation; it is expedient to put

$$A'_{x,y} f_{x,y} + \text{c.c.} = A'_{x,y} f_{x,y} + A^{*\prime}_{x,y} f^*_{x,y} = 0, \qquad (8)$$

so that

$$x' = A_x f_x' + \text{c.c.}, \qquad y' = A_y f_y' + \text{c.c.} \tag{9}$$

As a result of substitution (7), and using (4,9), Eqs. (1) are re-written for the complex amplitudes

$$\begin{aligned} f_x' A_x' + \text{c.c.} &= -(q + L'/2)\, y - L\, y' , \\ f_y' A_y' + \text{c.c.} &= -(q - L'/2)\, x + L\, x' . \end{aligned} \tag{10}$$

A serious simplification comes from our additional equation (8) and the normalization condition (5) applied successively on the left hand side of Eqs. (10):

$$f_x^* \left(f_x' A_x' + \text{c.c.} \right) = (f_x' f_x^* - f_x f_x^{*\prime}) A_x' = 2i A_x' .$$

Finally, we have:

$$\begin{aligned} A_x' &= \frac{i}{2} [(q + L'/2)\, y + L\, y']\, f_x^* , \\ A_y' &= \frac{i}{2} [(q - L'/2)\, x - L\, x']\, f_y^* , \end{aligned} \tag{11}$$

where x, x', y, y' on the right hand side should be substituted through their amplitudes and Floquet functions, using (7,9). Note that equations (11), together with their complex conjugate counterparts, form a complete set of four exact equations.

2.3 *Effect of a Skew-Quadrupole Kick*

Analyzing the terms on the RHS of (11), we recognize therein factors combined from the coupling coefficients and Twiss parameters which are periodic, and the exponential functions with running phases, e.g.:

$$\begin{aligned} & q y f_x^* = q (A_y f_y + \text{c.c.}) f_x^* = q \sqrt{\beta_x \beta_y} \\ & \times (A_y \exp(-i\psi_x + i\psi_y) + A_y^* \exp(-i\psi_x - i\psi_y)) \end{aligned} \tag{12}$$

When the lattice tunes are far from the sum and difference resonances, we have no reason to average out any terms and have to work with the complete set of four Eqs. (11).

We will consider skew-quadrupoles as short kicks placed at s_q: $q = \widehat{q} \delta(s - s_q)$. Integration of (11) over the kick range gives new amplitudes changed by the kick; we group the four amplitudes in a vector: $A^T = (A_x, A_x^*, A_y, A_y^*)$, and write the result of the kick action on initial amplitudes A_0 in the matrix form, $A = K\, A_0$, where the kick matrix K is given by

$$K = I + Q , \tag{13}$$

I here being the 4×4 identity matrix, and Q has a special structure and depends on two complex parameters a, b only:

$$Q = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b^* & a^* \\ -a^* & b & 0 & 0 \\ b^* & -a & 0 & 0 \end{pmatrix} . \tag{14}$$

These parameters are expressed via the kick strength and nominal lattice functions, using Eqs. (11-12)

$$\begin{aligned} a &= \tfrac{i}{2} \widehat{q}\, f_x^* f_y = ik \exp(-i\psi_x + i\psi_y) , && (15) \\ b &= \tfrac{i}{2} \widehat{q}\, f_x^* f_y^* = ik \exp(-i\psi_x - i\psi_y) . && (16) \end{aligned}$$

Here $k = \widehat{q} \sqrt{\beta_x \beta_y}/2$ stands for the normalized strength of the skew-quadrupole kick, and values of the lattice functions $\beta_{x,y}$, $\psi_{x,y}$ are taken at the kick location.

Having in mind that between the kicks vector A remains constant, combined action of N successive kicks is given by the matrix K_{tot} equal to the product of K_n ($0 \le n \le N$), where the lattice functions values in respective a_n, b_n are taken at the location of each kick.

2.4 *First-Order Approximation*

Concatenation of successive kicks results in accumulation of nonlinear terms in the matrix elements (i.e., products of a_n, b_n), complicating the analysis. A serious simplification comes from linearization of this problem which is applicable in the case of weak kicks $k_n \ll 1$. Then, to first order in the skew-quadrupole strengths, and using (13), we can write the total transformation

$$K_{tot} = \prod\nolimits_{n=0}^{n=N} K_n \approx I + \sum\nolimits_{n=0}^{n=N} Q_n . \tag{17}$$

Therefore, all we need to know is expressed via two complex numbers

$$a_{tot} = \sum\nolimits_{n=0}^{n=N} a_n , \qquad b_{tot} = \sum\nolimits_{n=0}^{n=N} b_n . \tag{18}$$

The main requirement to the desired combination of the quadrupoles is localized coupling. In terms of the amplitudes, A_x past this optics should have no contribution from A_y, and vice versa. Hence, in the matrix K_{tot} the off-diagonal 2×2 blocks must vanish. In the first-order approximation it means $a_{tot} = b_{tot} = 0$. For further use we rewrite two complex equations (18) explicitly, substituting from (15,16):

$$\begin{aligned} i a_{tot}^* &= \textstyle\sum_{n=0}^{n=N} k_n \exp(i\psi_{x,n} - i\psi_{y,n}) = 0, && (19) \\ i b_{tot}^* &= \textstyle\sum_{n=0}^{n=N} k_n \exp(i\psi_{x,n} + i\psi_{y,n}) = 0. && (20) \end{aligned}$$

Thus, we arrived at a very simple condition for localized coupling which is equivalent to *four equations in real variables* on N real parameters k_n. Provided that localization condition holds, we obtain the transparency condition $A = K_{tot} A_0 = I A_0 = A_0$ for free, in the first-order approximation in skew-quadrupole strengths.

Note that due to the off-diagonal form of matrix Q, see Eq. (14), the second order terms in $K_{tot} = \prod K_n = \prod (I + Q_n)$ will appear only in the diagonal blocks of K_{tot}, breaking the transparency (and resulting in tuneshifts $\Delta Q_{x,y}$ and $\beta_{x,y}$-beat in the outer lattice), but not affecting our localized coupling condition, since it is deduced from the off-diagonal blocks of K_{tot}. We conclude that Eqs. (19,20) actually provide coupling localization accurate to second order in skew-quadrupole strengths.

3 LOCALIZED COUPLING KNOB

3.1 Generic Solution

Having four homogeneous linear equations for localized coupling, we need in general five arbitrarily placed skew-quadrupoles to satisfy the equations with five unknown k_n while the locations of the kicks enter the coefficients of the linear equation set via the betatron phases. It is convenient to count the phase advances from the phase of the 0-th kick (hence there are eight phases), and to measure the kick strengths in units of k_0, i.e. we put $k_0 = 1$, $\psi_{x,0} = \psi_{y,0} = 0$ in (19-20). The resulting equations for four unknowns

$$1 + \sum_{n=1}^{n=4} k_n \exp(i\psi_{x,n} - i\psi_{y,n}) = 0 \qquad (21)$$

$$1 + \sum_{n=1}^{n=4} k_n \exp(i\psi_{x,n} + i\psi_{y,n}) = 0 \qquad (22)$$

(together with their complex conjugates) give a unique solution for $k_1..k_4$ provided that the system is non-degenerate. These long expressions are not presented here, numeric solution for the localized coupling knob components is readily available for each particular set of parameters. A similar approach can be found in Refs. 2–4.

Due to linear nature of the problem, having a few different knobs found, one may combine them in superposition.

The coefficients found from this first-order recipe may be refined on a general-purpose lattice-analysis code (e.g., SAD) in the range of k_n values intended for use.

3.2 Example of Coupling Control

Consider the local coupling control at point C, made with superposition of two knobs. As an example, we take two knobs so that point C is downstream the front skew-quadrupole of each of the two strings, while four skew-quadrupoles downstream point C are used to localize the coupling produced by each string.

We characterize point C by the values $\beta_{x,c}$, $\beta_{y,c}$ and $\psi_{x,c}$, $\psi_{y,c}$ taken at this point. In this section we will count the phases from those of the first upstream skew-quadrupole, belonging to knob 1, and take u_0 as its kick strength; then the next skew-quadrupole upstream point C belongs to knob 2 and has the values β_x, β_y and ψ_x, ψ_y at its location, and strength u.

Consider propagation of the horizontal mode from the outer lattice through this coupling control system. Its initial amplitude vector is $A_0^T = \frac{1}{2}(e^{i\phi}, e^{-i\phi}, 0, 0)$. From Eqs. (18,13-16) we obtain at point C in the first-order approximation $A_c = (\frac{1}{2}e^{i\phi}, \frac{1}{2}e^{-i\phi}, A_{y,c}, A_{y,c}^*)$ where

$$A_{y,c} = i\left(u_0 \cos\phi + u\cos(\phi + \psi_x)\exp(-i\psi_y)\right) \qquad (23)$$

According to the ansatz (7), and with $A_{x,0} = e^{i\phi}/2$, x-oscillation at point C is given by

$$\mathrm{Re}\{A_{x,c} f_{x,c}\} = \sqrt{\beta_{x,c}}\cos(\phi + \psi_{x,c}), \qquad (24)$$

and y-oscillation is

$$\mathrm{Re}\{A_{y,c} f_{y,c}\} = \sqrt{\beta_{y,c}}\mathrm{Re}\{A_{y,c}\exp(i\psi_{y,c})\}. \qquad (25)$$

Thus, the coupled oscillation of the horizontal mode forms a tilted ellipse at point C. We can choose u in (23) so as to tune either ellipticity or tilt independently.

To tune the tilt we put

$$u = -\frac{u_0 \sin\psi_{x,c}\sin\psi_{y,c}}{\sin(\psi_{x,c} - \psi_x)\sin(\psi_{y,c} - \psi_y)} \qquad (26)$$

to eliminate the ellipticity, then the tilt angle α is proportional to the strength of the first skew-quadrupole of knob 1,

$$\alpha = u_0 \sqrt{\frac{\beta_{y,c}}{\beta_{x,c}}} \frac{\sin\psi_x \sin\psi_{y,c}}{\sin(\psi_{x,c} - \psi_x)} \qquad (27)$$

To tune the ellipticity we put

$$u = -\frac{u_0 \cos\psi_{x,c}\sin\psi_{y,c}}{\cos(\psi_{x,c} - \psi_x)\sin(\psi_{y,c} - \psi_y)} \qquad (28)$$

thus eliminating the tilt, then the ratio of the vertical and horizontal axes of the ellipse r, proportional to the strength of the first skew-quadrupole of knob 1, is

$$r = u_0 \sqrt{\frac{\beta_{y,c}}{\beta_{x,c}}} \frac{\sin\psi_x \sin\psi_{y,c}}{\cos(\psi_{x,c} - \psi_x)} \qquad (29)$$

In our first-order approximation its strength is a small parameter, the coupling introduced by these knobs is weak, so $|\alpha|$, $r \ll 1$.

4 SUMMARY

A localized coupling control knob is available as combination of five skew-quadrupoles with arbitrary locations in the betatron $x - y$ phases. The first-order recipe may need refinement on a linear lattice analysis code, capable of full-coupling accounting. Lesser number of kicks in the localized coupling knob is possible at the expense of restrictive conditions imposed on their locations, which are unlikely to be met in a real machine, unless its lattice is originally designed so as to envisage for this option. The found knobs may be combined in superposition.

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REFERENCES

[1] A.A. Kolomensky and A.N. Lebedev. Theory of Cyclic Accelerators, North-Holland, 1966.

[2] G.Guignard, Preprint CERN 76-06, Geneva, 1976.

[3] G.Guignard, Preprint CERN SL/91-50(AP), Geneva, 1991.

[4] G.Guignard, Phys. Rev. **E 51** (1995),6104.