MULTIRESOLUTION REPRESENTATION FOR ORBITAL DYNAMICS IN MULTIPOLAR FIELDS

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Abstract

We present the applications of variation – wavelet analysis to polynomial/rational approximations for orbital motion in transverse plane for a single particle in a circular magnetic lattice in case when we take into account multipolar expansion up to an arbitrary finite number and additional kick terms. We reduce initial dynamical problem to the finite number (equal to the number of n-poles) of standard algebraical problems. We have the solution as a multiresolution (multiscales) expansion in the base of compactly supported wavelet basis.

1 INTRODUCTION

In this paper we consider the applications of a new numerical-analytical technique which is based on the methods of local nonlinear harmonic analysis or wavelet analysis to the orbital motion in transverse plane for a single particle in a circular magnetic lattice in case when we take into account multipolar expansion up to an arbitrary finite number and additional kick terms. We reduce initial dynamical problem to the finite number (equal to the number of n-poles) of standard algebraical problems and represent all dynamical variables as expansion in the bases of maximally localized in phase space functions (wavelet bases). Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and gives for the general type of operators (differential, integral, pseudodifferential) in such bases the maximum sparse forms. Our approach in this paper is based on the generalization of variational-wavelet approach from [1]-[8], which allows us to consider not only polynomial but rational type of nonlinearities [9]. The solution has the following form

$$z(t) = z_N^{slow}(t) + \sum_{j\geq N} z_j(\omega_j t), \quad \omega_j \sim 2^j \tag{1}$$

which corresponds to the full multiresolution expansion in all time scales. Formula (1) gives us expansion into a slow part z_N^{slow} and fast oscillating parts for arbitrary N. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first term in the RHS of equation (1) corresponds on the global level of function space decomposition to resolution space and the second one to detail space. In this way we give contribution to our full solution from each scale of resolution or each time scale. The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution. Starting in part 2 from Hamiltonian of orbital motion in magnetic lattice with additional kicks terms, we consider in part 3 variational formulation for dynamical system with rational nonlinearities and construct via multiresolution analysis explicit representation for all dynamical variables in the base of compactly supported wavelets.

2 PARTICLE IN THE MULTIPOLAR FIELD

The magnetic vector potential of a magnet with $2n$ poles in Cartesian coordinates is

$$A = \sum_n K_n f_n(x,y), \tag{2}$$

where f_n is a homogeneous function of x and y of order n. The real and imaginary parts of binomial expansion of

$$f_n(x,y) = (x+iy)^n \tag{3}$$

correspond to regular and skew multipoles. The cases $n = 2$ to $n = 5$ correspond to low-order multipoles: quadrupole, sextupole, octupole, decapole. The corresponding Hamiltonian ([10] for designation):

$$\begin{aligned} H(x,p_x,y,p_y,s) &= \frac{p_x^2+p_y^2}{2} + \\ \left(\frac{1}{\rho(s)^2} - k_1(s)\right)&\cdot\frac{x^2}{2} + k_1(s)\frac{y^2}{2} \\ -\mathcal{R}e&\left[\sum_{n\geq 2}\frac{k_n(s)+ij_n(s)}{(n+1)!}\cdot(x+iy)^{(n+1)}\right] \end{aligned} \tag{4}$$

Then we may take into account arbitrary but finite number of terms in expansion of RHS of Hamiltonian (4) and from our point of view the corresponding Hamiltonian equations of motions are not more than nonlinear ordinary differential equations with polynomial nonlinearities and variable coefficients. Also we may add the terms corresponding to kick type contributions of rf-cavity:

$$A_\tau = -\frac{L}{2\pi k}\cdot V_0\cdot\cos\left(k\frac{2\pi}{L}\tau\right)\cdot\delta(s-s_0) \tag{5}$$

or localized cavity $V(s) = V_0\cdot\delta_p(s-s_0)$ with $\delta_p(s-s_0) = \sum_{n=-\infty}^{n=+\infty}\delta(s-(s_0+n\cdot L))$ at position s_0. Fig.1 and Fig.2 present finite kick term model and the corresponding multiresolution representation on each level of resolution.

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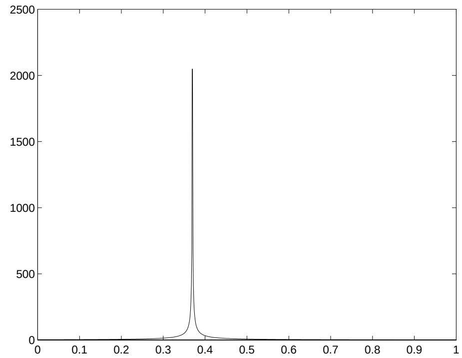

Figure 1: Finite kick model.

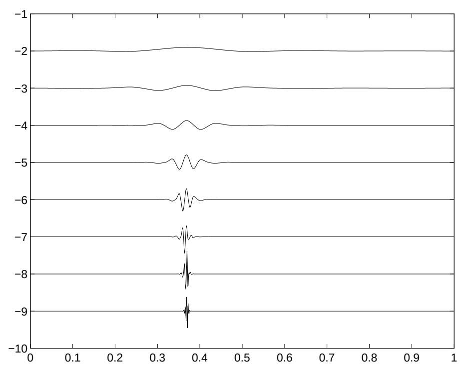

Figure 2: Multiresolution representation of kick.

3 RATIONAL DYNAMICS

The first main part of our consideration is some variational approach to this problem, which reduces initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. We have the solution in a compactly supported wavelet basis. Multiresolution expansion is the second main part of our construction. The solution is parameterized by solutions of two reduced algebraical problems, one is nonlinear and the second are some linear problems, which are obtained from one of the next wavelet constructions: the method of Connection Coefficients (CC), Stationary Subdivision Schemes (SSS).

3.1 Variational Method

Our problems may be formulated as the systems of ordinary differential equations

$$Q_i(x)\frac{\mathrm{d}x_i}{\mathrm{d}t} = P_i(x,t), \quad x = (x_1,...,x_n), \qquad (6)$$
$$i = 1,...,n, \quad \max_i deg\ P_i = p, \quad \max_i deg\ Q_i = q$$

with fixed initial conditions $x_i(0)$, where P_i, Q_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. Because of time dilation we can consider only next time interval: $0 \leq t \leq 1$.

Let us consider a set of functions

$$\Phi_i(t) = x_i\frac{\mathrm{d}}{\mathrm{d}t}(Q_i y_i) + P_i y_i \qquad (7)$$

and a set of functionals

$$F_i(x) = \int_0^1 \Phi_i(t)dt - Q_i x_i y_i\mid_0^1, \qquad (8)$$

where $y_i(t)$ $(y_i(0) = 0)$ are dual (variational) variables. It is obvious that the initial system and the system

$$F_i(x) = 0 \qquad (9)$$

are equivalent. Of course, we consider such $Q_i(x)$ which do not lead to the singular problem with $Q_i(x)$, when $t = 0$ or $t = 1$, i.e. $Q_i(x(0)), Q_i(x(1)) \neq \infty$.

Now we consider formal expansions for x_i, y_i:

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t) \quad y_j(t) = \sum_r \eta_j^r \varphi_r(t), \qquad (10)$$

where $\varphi_k(t)$ are useful basis functions of some functional space (L^2, L^p, Sobolev, etc) corresponding to concrete problem and because of initial conditions we need only $\varphi_k(0) = 0$, $r = 1,...,N, \quad i = 1,...,n$,

$$\lambda = \{\lambda_i\} = \{\lambda_i^r\} = (\lambda_i^1, \lambda_i^2, ..., \lambda_i^N), \qquad (11)$$

where the lower index i corresponds to expansion of dynamical variable with index i, i.e. x_i and the upper index r corresponds to the numbers of terms in the expansion of dynamical variables in the formal series. Then we put (10) into the functional equations (9) and as result we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of expansions (10):

$$L(Q_{ij}, \lambda, \alpha_I) = M(P_{ij}, \lambda, \beta_J), \qquad (12)$$

where operators L and M are algebraization of RHS and LHS of initial problem (6), where λ (11) are unknowns of reduced system of algebraical equations (RSAE)(12).

Q_{ij} are coefficients (with possible time dependence) of LHS of initial system of differential equations (6) and as consequence are coefficients of RSAE.

P_{ij} are coefficients (with possible time dependence) of RHS of initial system of differential equations (6) and as consequence are coefficients of RSAE.

$I = (i_1, ..., i_{q+2})$, $J = (j_1, ..., j_{p+1})$ are multiindexes, by which are labelled α_I and β_I — other coefficients of RSAE (12):

$$\beta_J = \{\beta_{j_1...j_{p+1}}\} = \int \prod_{1\leq j_k\leq p+1} \varphi_{j_k}, \qquad (13)$$

where p is the degree of polinomial operator P (6)

$$\alpha_I = \{\alpha_{i_1}...\alpha_{i_{q+2}}\} = \sum_{i_1,...,i_{q+2}} \int \varphi_{i_1}...\dot{\varphi}_{i_s}...\varphi_{i_{q+2}}, \qquad (14)$$

where q is the degree of polynomial operator Q (6), $i_\ell = (1, ..., q+2)$, $\dot{\varphi_{i_s}} = \mathrm{d}\varphi_{i_s}/\mathrm{d}t$.

Now, when we solve RSAE (12) and determine unknown coefficients from formal expansion (10) we therefore obtain the solution of our initial problem. It should be noted if we consider only truncated expansion (10) with N terms then we have from (12) the system of $N \times n$ algebraical equations with degree $\ell = max\{p, q\}$ and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (rational) problem in the form

$$x_i(t) = x_i(0) + \sum_{k=1}^{N} \lambda_i^k X_k(t), \qquad (15)$$

where coefficients λ_i^k are roots of the corresponding reduced algebraical (polynomial) problem RSAE (12). Consequently, we have a parametrization of solution of initial problem by solution of reduced algebraical problem (12). The first main problem is a problem of computations of coefficients α_I (14), β_J (13) of reduced algebraical system. These problems may be explicitly solved in wavelet approach.

Next we consider the construction of explicit time solution for our problem. The obtained solutions are given in the form (15), where $X_k(t)$ are basis functions and λ_k^i are roots of reduced system of equations. In our case $X_k(t)$ are obtained via multiresolution expansions and represented by compactly supported wavelets and λ_k^i are the roots of corresponding general polynomial system (12) with coefficients, which are given by CC or SSS constructions. According to the variational method to give the reduction from differential to algebraical system of equations we need compute the objects α_I and β_J [1],[9].

Our constructions are based on multiresolution approach. Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. We consider a multiresolution analysis of $L^2(\mathbf{R}^n)$ (of course, we may consider any different functional space) which is a sequence of increasing closed subspaces V_j:

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ... \qquad (16)$$

satisfying the following properties:

$$\bigcap_{j \in \mathbf{Z}} V_j = 0, \quad \overline{\bigcup_{j \in \mathbf{Z}}} V_j = L^2(\mathbf{R}^n),$$

On Fig.3 we present contributions to solution of initial problem from first 5 scales or levels of resolution.

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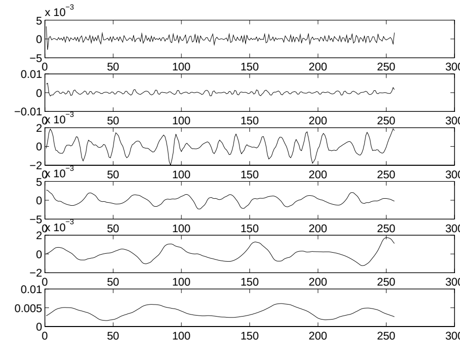

Figure 3: Contributions to approximation: from scale 2^1 to 2^5.

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