SCALING LAW FOR THE IMPACT OF MAGNET FRINGE FIELDS*

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Abstract

A general scaling law can be derived for the relative momentum deflection produced on a particle beam by fringe fields, to leading order. The formalism is applied to two concrete examples, for magnets having dipole and quadrupole symmetry.

1 INTRODUCTION

During recent years, the impact of magnet fringe fields is becoming increasingly important for rings of relatively small circumference but large acceptance. A few years ago, following some heuristic arguments, a scaling law was proposed [1], for the relative deflection of particles passing through a magnet fringe-field. In fact, after appropriate expansion of the magnetic fields in Cartesian coordinates, which generalizes the expansions of Steffen [2], one can show that this scaling law is true for any multipole magnet, at leading order in the transverse coefficients [3]. This paper intends to provide the scaling law to estimate the impact of fringe fields in the special cases of magnets with dipole and quadrupole symmetry.

2 GENERAL MULTIPOLE EXPANSION

The formalism presented here generalizes an approach described by Steffen and reduces to formulas he gives in the case of dipoles and quadrupoles [2]. After appropriate expansion of the magnetic scalar potential and the use of Laplace equation, one can show [3] that the magnetic field components can be written in a compact form as

$$
\begin{aligned}
B_x(x,y,z) &= \sum_{n,m=0}^{\infty}\sum_{l=0}^{m}(-1)^m\binom{m}{l}\frac{x^n\,y^{2m}}{n!\,(2m)!}\times \\
&\quad\left(b^{[2l]}_{n+2m+1-2l}\frac{y}{2m+1}+a^{[2l]}_{n+2m-2l}\right) \\
B_y(x,y,z) &= \sum_{n,m=0}^{\infty}\frac{(-1)^m x^n\,y^{2m}}{n!\,(2m)!}\left(\sum_{l=0}^{m}\binom{m}{l}b^{[2l]}_{n+2m-2l}\right. \\
&\quad\left.-\sum_{l=0}^{m+1}\binom{m+1}{l}a^{[2l]}_{n+2m+1-2l}\frac{y}{2m+1}\right), \\
B_z(x,y,z) &= \sum_{n,m=0}^{\infty}\sum_{l=0}^{m}(-1)^m\binom{m}{l}\frac{x^n\,y^{2m}}{n!\,(2m)!}\times \\
&\quad\left(b^{[2l+1]}_{n+2m-2l}\frac{y}{2m+1}+a^{[2l+1]}_{n+2m-1-2l}\right)
\end{aligned}
\tag{1}
$$

In an idealized model of a magnet, only one (or in the case of combined function magnets, two) of the multipole coefficients will be non-vanishing in the body of the magnet (length $L_{\rm eff}$) and in this region only the $l=0$ terms in the expansions survive. One can make many useful remarks about symmetries of the skew and normal multipole coefficient in a quite straightforward way through these expressions.

3 DIPOLE FRINGE FIELD

Continuing to ignore bending of the centerline in a dipole magnet, the configuration of poles and coils is symmetric about the $x=0$ and $y=0$ planes, and the coils are excited with alternating signs and equal strength. Initially, to illustrate the treatment of allowed multipoles, we will permit the magnet to be not quite ideal but with coils that respect the dipole magnet symmetries. For this to be true, the magnetic field will satisfy the following symmetry conditions: B_x is odd in x and odd in y; B_y is even in both x and y; B_z is even in x and odd in y. Using the general field expansion of Eq. (1), we get:

$$
\begin{aligned}
B_x &= \sum_{m,n=0}^{\infty}\sum_{l=0}^{m}\frac{(-1)^m x^{2n+1}y^{2m+1}}{(2n+1)!(2m+1)!}\binom{m}{l}b^{[2l]}_{2n+2m+2-2l} \\
B_y &= \sum_{m,n=0}^{\infty}\sum_{l=0}^{m}\frac{(-1)^m x^{2n}y^{2m}}{(2n)!(2m)!}\binom{m}{l}b^{[2l]}_{2n+2m-2l} \\
B_z &= \sum_{m,n=0}^{\infty}\sum_{l=0}^{m}\frac{(-1)^m x^{2n}y^{2m+1}}{(2n)!(2m+1)!}\binom{m}{l}b^{[2l+1]}_{2n+2m-2l}
\end{aligned}
$$

Taking the field expansion up to leading order, we get:

$$
\begin{aligned}
B_x &= b_2xy+O(4) \\
B_y &= b_0-\frac{1}{2}b_0^{[2]}y^2+\frac{1}{2}b_2(x^2-y^2)+O(4) \\
B_z &= y\;b_0^{[1]}+O(3)
\end{aligned}
\tag{2}
$$

where b_2 represents a sextupole field component allowed by the symmetry of the "dipole" magnet (for an ideally designed magnet $b_2=0$) and $O(3)$ and $O(4)$ contain all the allowed terms of higher orders.

For a particle traversing the magnet with a horizontal deviation x and vertical deviation y from the center, the impulse (i.e. change of transverse momentum) imparted by the nominal field gradient is

$$\Delta p_0=-e\int v_zb_0(0)dz\approx -ev_zb_0(0)L \tag{3}$$

where $L=\int b_0dz/b_0(0)$ is the effective length of the magnet, and $b_0(0)$ is the dipole coefficient in the body of the "dipole" magnet.

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The impulse due to the fringe field at one end of a magnet is defined in this paper as the effect of field deviation from nominal, from well inside (where the nominal multipole coefficient is assumed to be independent of z) to well outside the magnet (where all field components are assumed to vanish.) These will be the limits for subsequent integrals. To obtain explicit formulas the upper limit of these integrals will be taken to be infinity. Exploiting the assumed constancy of x and y along the orbit, these integrals will all be evaluated using integration by parts. Suppressing the entire pure dipole contribution, we have $\int_{-\infty}^{\infty} \mathbf{B}(x,y,z)dz \approx 0$. For $x = y = 0$ this is an equality *by definition*, and for finite displacements it is approximately true if (as we are assuming) the transverse particle displacements remain approximately constant. At each magnet end, the relative change of particle position across the fringe region is typically much smaller than the relative change of field strength; i.e. $|\beta'_{x,y}/\beta_{x,y}| \ll |b_0^{[1]}/b_0|$. The momentum increments of the particle caused by the longitudinal component of the magnetic field are given by

$$\begin{aligned} \Delta p_x(\|) &= e\int v_z y' B_z dz \approx e v_z b_0 y y' \\ \Delta p_y(\|) &= -e\int v_z x' B_z dz \approx -e v_z b_0 y x' \end{aligned} \quad . \tag{4}$$

The momentum increments caused by the transverse component of the fringe fields are

$$\begin{aligned} \Delta p_x(\perp) &= -e\int v_z B_y dz \approx e v_z b_0 y y' \\ \Delta p_y(\perp) &= e\int v_z B_x dz \approx 0 \end{aligned} \quad . \tag{5}$$

The total momentum increments due to fringe field are therefore

$$\begin{aligned} \Delta p_x &\approx 2 e v_z b_0 y y' \\ \Delta p_y &\approx -e v_z b_0 y x' \end{aligned} \quad . \tag{6}$$

The factors xx', yy', xy', and $x'y$ can be averaged as follows. By the standard "pseudo-harmonic" description of betatron motion, letting $S_{x,y} = \sin\psi_{x,y}$, $C_{x,y} = \cos\psi_{x,y}$,

$$x = \sqrt{\epsilon_x \beta_x}\, C_x, \quad x' = \sqrt{\frac{\epsilon_x}{\beta_x}}(S_x - C_x \frac{\beta'_x}{2}), \tag{7}$$

and the same for y, y'. Using the results $\langle C^2_{x,y}\rangle = \langle S^2_{x,y}\rangle = 1/2$, $\langle C^2_{x,y} S^2_{x,y}\rangle = 1/8$, $\langle C^4_{x,y}\rangle = 3/8$, and $\langle C^3_{x,y} S_{x,y}\rangle = 0$, and assuming uncorrelated x and y motion, one obtains

$$\begin{aligned} \langle x^2 {x'}^2\rangle &= \frac{\epsilon_x^2 f_1(\beta'_x)}{8}, & \langle y^2 {y'}^2\rangle &= \frac{\epsilon_y^2 f_1(\beta'_y)}{8}, \\ \langle y^2 {x'}^2\rangle &= \frac{\epsilon_x\epsilon_y f_2(\beta'_x)\beta_y}{4\beta_x}, & \langle x^2 {y'}^2\rangle &= \frac{\epsilon_x\epsilon_y f_2(\beta'_y)\beta_x}{4\beta_y}, \end{aligned} \tag{8}$$

where $f_1(\beta') = 1 + 3{\beta'}^2/4$, $f_2(\beta') = 1 + {\beta'}^2/4$, and $\beta' = d\beta/dz$ and the symbol $\langle\ \rangle$ denotes the average over betatron phase. The ratio between the momentum increment produced by the fringe field to that produced by the non-fringe field is approximately

$$\begin{aligned} \frac{\langle|\Delta p_x|\rangle}{\langle|\Delta p_0|\rangle} &\approx \frac{\epsilon_y\sqrt{f_1(\beta'_y)}}{\sqrt{2}\,L} \\ \frac{\langle|\Delta p_y|\rangle}{\langle|\Delta p_0|\rangle} &\approx \frac{\sqrt{\epsilon_x\epsilon_y\beta_y f_2(\beta'_x)}}{2\sqrt{\beta_x}L} \end{aligned} \quad . \tag{9}$$

For magnets in non-critical locations (which is to say most magnets) the values of f_1 and f_2 are in the range from 1 to 2, so a "back of the envelope" estimate of the impulse is given by

$$\frac{\langle|\Delta p_\perp|\rangle}{\langle|\Delta p_0|\rangle} \approx \frac{\epsilon_\perp}{L} \quad , \tag{10}$$

where $\epsilon_\perp$ is the rms beam transverse emittance. Often this ratio is so small as to make neglect of the fringe field deflections entirely persuasive. The simplicity of the formula is due to the fact that the fringe contribution is expressed as a fraction of the dominant contribution. Note that, as stated before, this formula applies to each end separately, and does not depend on any cancelation of the contribution from two ends. The case in which fringe deflections are likely to be most important is when β'_x or β'_y is anomalously large, for example in the vicinity of beam waists such as at the location of intersection points in colliding beam lattices. In this case, and by just keeping the dominant term of (9) the deflections can be approximated by

$$\frac{\langle|\Delta p_\perp|\rangle}{\langle|\Delta p_0|\rangle} \approx \frac{\epsilon_\perp}{L}\beta'_{max} \quad , \tag{11}$$

where β'_{max} stands either for β'_x or β'_y.

4 QUADRUPOLE FRINGE FIELD

The configuration of poles and coils in a quadrupole magnet is symmetric about the four planes $x = 0; y = 0;\ x = y;\ x = -y$ and if the coils are excited with alternating signs and equal strength, the magnetic field will satisfy the following symmetry conditions: B_x is even in x and odd in y; B_y is odd in x and even in y; B_z is odd in both x and y; and $B_z(x,y,z) = B_z(y,x,z)$. As before, we may express the field components as:

$$\begin{aligned} B_x &= \sum_{m,n=0}^{\infty}\sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!}\binom{m}{l} b^{[2l]}_{2n+2m+1-2l} \\ B_y &= \sum_{m,n=0}^{\infty}\sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m}}{(2n+1)!(2m)!}\binom{m}{l} b^{[2l]}_{2n+2m+1-2l} \\ B_z &= \sum_{m,n=0}^{\infty}\sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!}\binom{m}{l} b^{[2l+1]}_{2n+2m+1-2l} \end{aligned} \quad .$$

The field expansion can be written as

$$
\begin{aligned}
B_x &= y\left[b_1 - \frac{1}{12}(3x^2+y^2)b_1^{[2]}\right] + O(5) \\
B_y &= x\left[b_1 - \frac{1}{12}(3y^2+x^2)b_1^{[2]}\right] + O(5) \\
B_z &= xyb_1^{[1]} + O(4)
\end{aligned} \tag{12}
$$

where $b_1(z) = \frac{\partial B_x}{\partial y}|_{x=y=0} = \frac{\partial B_y}{\partial x}|_{x=y=0}$ is the transverse field gradient at the quadrupole axis, and $O(4), O(5)$ contain all the higher order terms. Note also that $b_3 = -b_1^{[2]}/2$, due to the field symmetry. For quadrupoles, common notations are $g(z) \equiv b_1(z)$ and $g_0 \equiv g(0) \equiv b_1(0)$. For a particle traversing the magnet with a horizontal deviation x and vertical deviation y from the center, the momentum increments produced by the nominal field gradient are

$$
\Delta p_{x0} = -ev_z g_0 x L, \quad \Delta p_{y0} = ev_z g_0 y L \tag{13}
$$

where$L = \int g(z)dz/g_0$ is the effective length of the magnet, and g_0 is the gradient in the body of the quadrupole magnet. Similar to the case of dipole magnet, suppressing the quadrupole contribution from $\mathbf{B}$, one obtains $\int_{-\infty}^{\infty} \mathbf{B}(x,y,z)dz \approx 0$ i.e. the integrated effect of the longitudinal component of the fringe field across the entire quadrupole magnet is small if (as assumed throughout) the particle transverse displacements remain approximately constant. At each magnet end across the fringe field region, the relative change of particle position is typically much smaller than the relative change of field strength, i.e. $|\beta'_{x,y}/\beta_{x,y}| \ll |g'_0/g_0|$. The momentum increments of the particle contributed from the longitudinal component of the magnetic field are

$$
\begin{aligned}
\Delta p_x(\|) &= e\int v_z y' B_z dz \approx ev_z xyy' g_0 \\
\Delta p_y(\|) &= -e\int v_z x' B_z dz \approx -ev_z xyx' g_0
\end{aligned} \; . \tag{14}
$$

The momentum increment produced by the transverse component of the fringe fields are

$$
\begin{aligned}
\Delta p_x(\perp) &= -ev_z\frac{1}{2}xyy'g_0 - ev_z\frac{1}{4}(x^2+y^2)x'g_0 \\
\Delta p_y(\perp) &= ev_z\frac{1}{2}xyx'g_0 + ev_z\frac{1}{4}(x^2+y^2)y'g_0
\end{aligned} \; . \tag{15}
$$

Combining the contributions, the total momentum increments due to fringe field are

$$
\begin{aligned}
\Delta p_x \approx & ev_z\frac{1}{2}xyy'g_0 - ev_z\frac{1}{4}(x^2+y^2)x'g_0 \\
\Delta p_y \approx & -ev_z\frac{1}{2}xyx'g_0 + ev_z\frac{1}{4}(x^2+y^2)y'g_0
\end{aligned} \; . \tag{16}
$$

Again, by using the standard "pseudo-harmonic" description of betatron motion and assuming uncorrelated x and y motion, one obtains the ratio between the momentum increment produced by the fringe field to that produced by the non-fringe field as

$$
\begin{aligned}
\frac{\langle|\Delta p_x|\rangle}{\langle|\Delta p_{x0}|\rangle} \approx & \frac{\beta_x}{8\sqrt{2}\bar{\beta}_x L}\left\{\epsilon_x\epsilon_y\left[-3\beta'_x\beta'_y + \frac{2\beta_y f_1(\beta'_x)}{\beta_x}\right] + \right. \\
& \left. \epsilon_x^2 f_3(\beta'_x) + \epsilon_y^2\left[4f_1(\beta'_y) + \frac{3\beta_y^2 f_1(\beta'_y)}{\beta_x^2} - \frac{3\beta_y\beta'_x\beta'_y}{\beta_x}\right]\right\}^{1/2} \\
\frac{\langle|\Delta p_y|\rangle}{\langle|\Delta p_{y0}|\rangle} \approx & \frac{\beta_y}{8\sqrt{2}\bar{\beta}_y L}\left\{\epsilon_x\epsilon_y\left[-3\beta'_x\beta'_y + \frac{2\beta_x f_1(\beta'_y)}{\beta_y}\right] + \right. \\
& \left. \epsilon_y^2 f_3(\beta'_y) + \epsilon_x^2\left[4f_1(\beta'_x) + \frac{3\beta_x^2 f_1(\beta'_x)}{\beta_y^2} - \frac{3\beta_x\beta'_x\beta'_y}{\beta_y}\right]\right\}^{1/2}
\end{aligned} \tag{17}
$$

where $f_3(\beta') = 1 + 5\beta'^2/4$ and the bar over the β functions denotes their average on the body of the magnet.

For quadrupoles in non-critical locations, the same assumptions as were made for dipoles yields "back of the envelope" estimate

$$
\frac{\langle|\Delta p_\perp|\rangle}{\langle|\Delta p_0|\rangle} \approx \frac{\epsilon_\perp}{L} \; . \tag{18}
$$

The same estimate is therefore applicable to both erect dipole and erect quadrupole deflections. As was true for dipoles, the fringe fields of quadrupoles become most important near beam waists where the β's are large. In that case, the fractional deflections become, as before,

$$
\frac{\langle|\Delta p_\perp|\rangle}{\langle|\Delta p_0|\rangle} \approx \frac{\epsilon_\perp}{L}\beta'_{max} \; , \tag{19}
$$

where β'_{max} stands either for β'_x or β'_y. Remarkably, it can be shown that this estimate is true for every multipole magnet [3].

5 CONCLUSION

We have shown in two concrete examples that the relative momentum deflection due to magnet fringe-fields, to leading order, is proportional to the transverse emittance and inversely proportional to the effective length of the magnet, in cases where the magnet is not in a critical location, i.e there is no violent variation of the optical functions. If the above is not true, the scaling is modified by just a factor of the maximum β'. These scaling laws are in agreement with previous estimations [1] and can be proved for any multipole magnet [3].

6 REFERENCES

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