PHASE-SPACE ANALYSIS BY MULTIPLE RESONANCE-FREQUENCY IDENTIFICATION: APPLICATIONS TO THE LHC AND LEP

R.T. Burgess* J.M. Jowett CERN, Geneva, Switzerland

Abstract

Many beam-dynamical phenomena are studied, experimentally or computationally, by means of spectral analysis of a time-series of values of a dynamical variable. When the underlying dynamics is regular, the frequencies appearing in the spectrum are integer combinations of a small set of basic frequencies, e.g., the three tunes in the case of single-particle orbital dynamics. For well-known reasons, identification of the frequencies can be ambiguous or subjective in practice.

We present an algorithm that overcomes these difficulties by exploiting theoretical bounds on the spectral power density to transform time series into sets of labelled resonance lines. In our examples, the time series are orbits obtained by tracking single particles from many initial conditions.

The method has been applied to off-momentum LHC injection optics. This is a deterministic Hamiltonian system. A second application, to orbits with strong quantum fluctuations in LEP2, shows that it also works well in a noisy, dissipative system. *

1 INTRODUCTION

Anyone who has looked at the tune spectrum of a stored beam will appreciate that unambiguous identification of the lines that appear is not always easy and may be partly subjective. In this paper, we develop an algorithm to solve this problem (in certain conditions) and apply it to particle tracking data.

Our multiple resonance frequency identification (MRFI) algorithm exploits bounds on the spectral power density given by KAM theory [1,2] in an heuristic manner, to return quantitative evaluations of the integer combinations, so identifying the spectral lines. A preliminary version of the algorithm was used in [3]; related theoretical background and methods are discussed in [4].

2 PRINCIPLES OF MRFI

The discrete Fourier transform maps a time-series of N values of a dynamical variable onto a spectrum over a set of frequencies $\left(0, \frac{1}{N}, \frac{2}{N}, \cdots, \frac{1}{2}\right)$. In the following, we mainly consider the corresponding *power* spectrum.

The MRFI algorithm has two parts: first the spectrum of a dynamical variable is transformed into a set of peaks (frequencies, amplitudes and possibly widths); secondly, each peak is assigned to one of the resonance frequencies. The appropriate peak-identification method depends on the nature of the dynamical system. Two illustrative cases are treated below.

The second part of the algorithm is motivated by results from KAM theory. When the underlying dynamics of a system is *quasi-periodic*, it is well known that the frequencies appearing in the spectrum are integer combinations of a small set of basic frequencies. For particle motion in an accelerator (3 degrees of freedom), the basic frequencies (in "tune" units) are $\mathbf{q} = (q_1, q_2, q_3)$, and peaks appear at the integer combinations $\mathbf{k}.\mathbf{q}$, $\mathbf{k} = (k_1, k_2, k_3) \in \mathbf{Z}^3$. KAM theory [1,2] suggests the definition of a function:

$$c(f, \mathbf{k}, \mathbf{q}) = \|\mathbf{k}\|^4 \min\left(|f - \{\mathbf{k}\cdot\mathbf{q}\}|, |f - 1 + \{\mathbf{k}\cdot\mathbf{q}\}|\right) \quad (1)$$

Here, the resonance order $\|\boldsymbol{k}\| = |k_1| + |k_2| + |k_3|$ and $\{x\}$ denotes the fractional part of x. Moreover, the amplitudes of the corresponding spectral components are bounded by $Me^{-\lambda\|\mathbf{k}\|}$ for some constants $M, \lambda > 0$.

Knowing the tune **q,** the function c is a measure of the distance from the frequency of some peak, f, to the resonance line labelled by **k.** This distance is minimised over some $\mathbf{K}(r) \subseteq \{\mathbf{k} \in \mathbf{Z}^3 : \|\mathbf{k}\| \le r\}$. (The inclusion, $\subseteq$, allows for the possible application of a selection rule within the octahedral set of all resonances up to the order r.) The minimum gives the resonance $\mathbf{k}^*$ most likely to have generated the peak:

$$c(f, \mathbf{k}^*, \mathbf{q}) = C(f, \mathbf{q}) = \min_{\mathbf{k}\in\mathbf{K}(r)} c(f, \mathbf{k}, \mathbf{q}) \quad (2)$$

For illustration, Figure 1 plots $C(f, \mathbf{q})$, for $\mathbf{q} = (0.35, 0.2, 0.12)$ and $\mathbf{K}(2) = \{\mathbf{k} \in \mathbf{Z}^3 : \|\mathbf{k}\| \le 2\}$, i.e., all resonances up to the second order.

The first order resonances are visible as minima in $C(f, \mathbf{q})$ at $f = 0.12, 0.2, 0.35$. The comparatively weaker influence of the second order resonances is indicated by the nine much narrower troughs at the positions of each resonance.

3 LHC AT INJECTION

The LHC injection optics as a function of the central beam momentum is well corrected for shifts in central momentum of ± 0.002 [7]. Hence a scan of phase space presents few peaks within the Fourier spectra of particles. To produce a more complex phase space and show the need for the b_5 correctors, we consider the example of the LHC optics V6.0 with b_5 correctors switched off. In our

* Technical student from the University of Bath, UK.

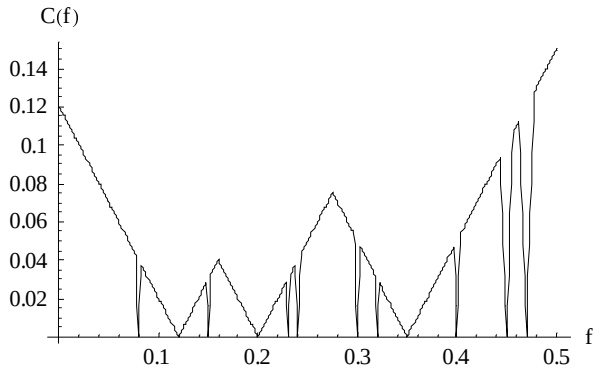

Figure 1: The distance C to 1st and 2nd order resonances.

example, particles were tracked in 4D using MAD8 [5] for $N = 1000$ turns with a shift in central beam momentum of +0.002. We illustrate the application of MRFI to the spectrum of the *normalised* co-ordinate x_n.

Spectral peaks are first identified to accuracy $1/N$ in the Fourier spectra. Since the system is deterministic, their frequencies can be resolved more accurately with the help of the NAFF method [6]; for quasi-periodic data the intrinsic error is $\propto N^{-4}$ and the MRFI method then works better.

Initially, single particle spectra were investigated. At small-to-moderate amplitudes, the two tunes were taken to be near the frequencies of highest peak occurred in the Fourier spectra of the x_n, y_n co-ordinates; see Figure 2.

The frequency and amplitude of peaks found by the algorithm match the original spectrum well. In a well-corrected machine spectra would exhibit only one main peak, with possibly another of very small amplitude.

To study the appearance and disappearance of particular resonances along a line in phase space, a set of particles with initial action $I_x = 7.82\times10^{-10}$ m, and increasing I_y, were tracked through the above-mentioned optics. The peaks and associated $\boldsymbol{k}$ for each particle were calculated as before.

At large initial I_y other peaks can be greater in magnitude than the "tune" peak. To ensure that the correct frequencies were attributed to the tunes, the values found by the algorithm were compared with those evaluated for the previous particle and the identity of the principal tune peak retained by continuity as far as reasonably possible.

Figure 3 tracks all resonances up to the 6th order as the initial action I_y co-ordinate is increased; the comparison with the background spectral density shows that MRFI provides a good quantitative representation of the main features.

Particular resonances may also be tracked through a plane in phase space. As before, it is necessary to track the tunes from the smallest amplitude outwards whilst performing MRFI on all particles that survived 1000 turns within that plane of phase space.

A set of MRFI plots, each depicting the amplitude of a different $\boldsymbol{k}$ found, at all points in phase space, produce a

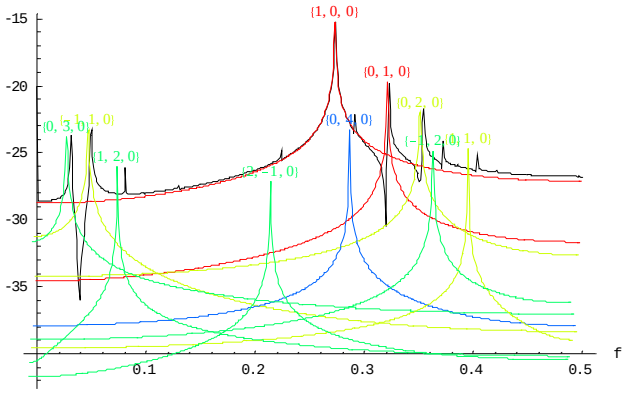

Figure 2 (colour): The original spectrum (black), the resulting peaks and associated k, coloured by their resonance order for a single particle with initial actions $I_x = 7.82\times10^{-10}$ m, $I_y = 1.27\times10^{-7}$ m.

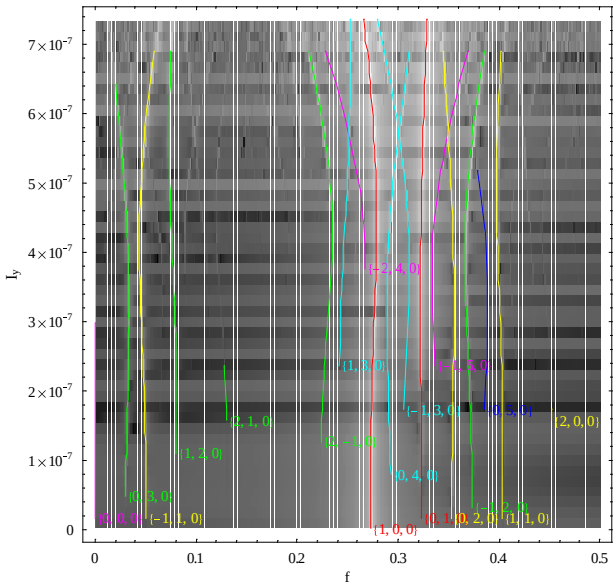

Figure 3 (colour): Tracking various resonances on the background power spectra (lighter corresponds to higher spectral density) as the action I_y, is varied. The resonances are overlaid as labelled lines, coloured to indicate the resonance order.

complete picture of the resonances affecting any region of phase space tracked.

A combination of 3 MRFI plots, for $\boldsymbol{k}$ =(0,3,0), (2,1,0) and (5,0,0) in the I_x-I_y plane is shown in Figure 4. Each colour denotes one $\boldsymbol{k}$. The shading of the plot represents the amplitude of the resonant peak at that point in phase space. In this case, the fainter the shade on the plot the smaller the amplitude of the resonance at that point relative to the highest amplitude measured for that particular resonance.

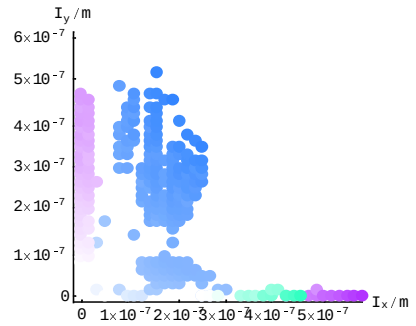

Figure 4 (colour): The resonances (0,3,0), (2,1,0) and (5,0,0) in I_x - I_y phase space (in purple, blue and green respectively.)

Thus, MRFI provides an additional tool to map out the features of the phase space of a Hamiltonian system.

4 LEP2

Single-particle motion in LEP2 is dominated by synchrotron radiation whose quantum fluctuations introduce a strong stochastic element [8,3]. The NAFF algorithm is not applicable to this noisy and dissipative system. Nevertheless the orbit may contain information about resonances present in the underlying Hamiltonian system (that includes the non-dissipative part of the radiation defining the closed orbit). To apply MRFI we need a different peak-finding algorithm.

Compared to the smooth spectra exhibited by the LHC model, the power spectra of a particle's displacement from the closed orbit contains much more noise. The application of a "moving average" such that every point is replaced by the average position over some *n* surrounding points reduces this level of noise and the true spectral peaks emerge more clearly.

Our method works by sorting the points within the spectrum in order of power density. The first point in this list is taken to be the tip of the first peak; subsequent points are tested in turn to see whether they lie within a certain frequency range of the peak. If so they are taken to be included within that peak; otherwise they are taken to be the tips of new peaks. This continues until all data has been sorted, leaving a list of peak objects, with specific frequencies, heights and widths. These peaks may then be analysed using the MRFI method described in Section 2.

This procedure has been applied to a number of different tracking cases in LEP2. As an example, Figure 5 shows the analysis of the Fourier spectrum of the normalised y_n co-ordinate of a particle started on the closed orbit and tracked for 10^4 turns (some 200 damping times). The beam energy was 96 GeV, the phase advances per cell were $(\mu_x, \mu_y) = (102°, 90°)$. After simulation of typical operational correction procedures, the particular imperfect machine chosen had a vertical emittance of $\varepsilon_y = 0.78$ nm and a well-corrected vertical dispersion (RMS $D_y = 0.03$ m).

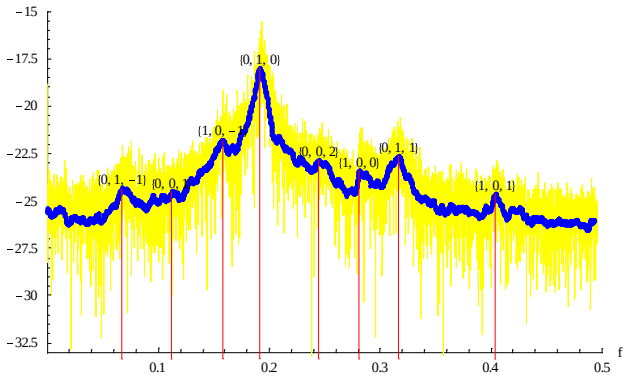

Figure 5 (colour): Analysis of a LEP2 positron spectrum with quantum fluctuations.

The MRFI method was applied to find resonances up to fifth order, using a 50 point moving average. The original spectrum, its moving average, and the labelled resonance lines produced by the algorithm are depicted in yellow, blue and red respectively.

The labelled resonances are in good agreement with the peaks in the moving average of the spectrum. Compared with the cases shown in [3] the absence of second-order synchro-betatron sidebands is consistent with the small vertical dispersion and relatively small vertical emittance. MRFI remains applicable despite the loss in accuracy due to the intrinsic nature of the system that requires the different peak finding method and the moving average.

5 CONCLUSIONS

The multiple resonance frequency identification (MRFI) method is a useful quantitative tool for detailed insight into the resonance terms influencing particle motion throughout phase space. Unlike some other methods, it is applicable both to deterministic Hamiltonian systems such as tracked orbits in a proton storage ring and to noisy dissipative systems like electron orbits with quantum fluctuations. It could, of course, also be applied to other problems inside and outside the accelerator physics context. The algorithms are implemented in Mathematica packages, available from [9].

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