OPTIMUM OPTICAL SYSTEMS

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Abstract

Using initial conditions derived from the thin lens approximation, the optimization of optical modules with finite length elements converges towards an absolute minimum. The safe determination of the initial conditions and of the constraints, using subsets of independent variables, depends on the optical properties of each module, and is a prerequisite to the automatic elaboration of optical modules made of real elements.

1 INTRODUCTION

The numerical optimization of an n-variable function is strongly dependent on initial conditions. In the case of particle beam optics, a model based on thin lenses provides one or several families of solutions which can be determined analytically. Then, these solutions are used as initial conditions for the numerical optimization, which safely converges towards a single minimum in the neighbourhood of the initial condition. The final solutions correspond to ”optimum” optical systems.

In the first section, the basics of symbolic optics [1] are recalled and the general method is described. It is then applied to *BeamOptics* [2], a symbolic program based on *Mathematica* [3] with two different examples: the isochronous period and the doublet used for betatron matching.

2 SYMBOLIC APPROACH TO OPTIMUM OPTICAL SYSTEMS

2.1 Symbolic Optics

In the framework of paraxial optics, the mapping of coordinates in the transverse plane from the input to the output of an optical element is linear and characterized by transfer matrices. Three types of objects are considered: drift spaces, bending magnets and quadrupoles. For instance, in a curvilinear system, a quadrupole of length l is rendered by the matrix:

$$M = \begin{pmatrix} \cos\sqrt{kl} & \frac{\sin\sqrt{kl}}{\sqrt{k}} & 0 \\ -\sqrt{k}\sin\sqrt{kl} & \cos\sqrt{kl} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

where the strength k is positive if the quadrupole is focusing in that plane, and negative otherwise. The third component refers to off-momentum particles and is used to compute the dispersion and its derivative. The betatron motion is described by the upper 2×2 matrix. The phase-amplitude formalism of Courant-Snyder theory [4] is based on the solution of the equation of motion in polar coordinates and describes the betatron motion through the optical α, β and γ functions gathered in the 2×2 σ matrix:

$$\sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \quad (2)$$

where α, β, γ are the characteristic lattice functions.

Tracing β- and α-functions through an optical element i of transfer matrix M_i results from the recursion of the σ matrix:

$$\sigma_i = M_i \sigma_{i-1} M_i^t. \quad (3)$$

Modules are built by assembling various basic elements (quadrupoles, drifts, bending magnets). The parameters in the matrix are the physical properties of the magnetic elements (length, strength, bending angle). In the thin lens model, equations are algebraic and exact solutions are found. These solutions are the initial conditions for the optimization described in the following.

2.2 Optimum Optical Systems Method

The procedure towards the automatic elaboration of thick lens structures using the optimum optical sytems method is given below. The first step is to split a machine into smaller independent modules to be treated separately, and then for each module:

- Build an analytical channel made of real, finite length lenses
- Derive an analytical, quadratic, χ^2-type function for minimization, including all the parameters of the module
- Choose the variables of the function
- Compute the algebraic initial values given by the thin lens model
- Minimize the function

One major point is the choice of the analytical function to be minimized. Following a χ^2 approach, different quadratic functions are built, according to the constraints of the structure. Indeed, for each type of module, there are initial or cyclic boundary conditions for the vertical and horizontal motions. Betatron matching is made either between waist conditions:

$$\beta_h = \beta_v \quad \text{and} \quad \alpha_h = \alpha_v = 0 \quad (4)$$

or crossover conditions:

$$\beta_h = \beta_v \quad \text{and} \quad \alpha_h = -\alpha_v \quad (5)$$

whereas periodic modules require periodic betatron and dispersion functions over the cell length L:

$$\beta_{h,v}(s+L) = \beta_{h,v}(s) \quad , \quad \alpha_{h,v}(s+L) = \alpha_{h,v}(s) \tag{6}$$
$$D(s+L) = D(s) \quad , \quad D'(s+L) = D'(s) \tag{7}$$

where D is the dispersion function and D' its derivative. In both cases, the difference between a thin and a finite length element is considered as a first-order perturbation. The goal is to minimize the error vector induced by this perturbation in the motion of particles. When dealing with modules made of quadrupoles, a quadrupolar perturbation is considered. It affects the beam envelope and the mean value of the phase variation. In the case of matching modules, the error vector on the betatron functions is given in the transverse plane by:

$$\delta = \begin{pmatrix} \frac{\beta_{h,v}-\beta^*_{h,v}}{\beta^*_{h,v}} \\ \alpha^*_{h,v}\frac{\beta_{h,v}-\beta^*_{h,v}}{\beta^*_{h,v}} - (\alpha_{h,v}-\alpha^*_{h,v}) \end{pmatrix} \tag{8}$$

where functions with a * superscript are the target functions, obeying either conditions (4) or (5). The function to minimize is then

$$U = (\delta_1)^2 + (\delta_2)^2 = (\frac{\Delta\beta}{\beta^*})^2 + (\alpha^*\frac{\Delta\beta}{\beta^*} - \Delta\alpha)^2. \tag{9}$$

In the case of periodic modules, the sum of the phase perturbations produced by each quadrupolar kick gives a tune shift

$$\Delta Q = \frac{1}{4\pi}\sum_j \beta_j(\Delta kl)_j \tag{10}$$

and the function to minimize is therefore

$$U = (\mu_h - \mu_h^*)^2 + (\mu_v - \mu_v^*)^2 \tag{11}$$

where μ is the betatron phase advance. For modules in which the dispersion curve has specific properties, the dispersion function D has to be computed since it is influenced by the long dipoles. This is the case of the isochronous periods for which equation (11) is slightly modified into

$$U = \alpha^2 + (\mu_v - \mu_v^*)^2 \quad \text{where} \quad \alpha = \frac{1}{L}\int_0^L \frac{D}{\rho}ds \tag{12}$$

is the momentum compaction computed analytically over the whole cell of length L, and ρ is the radius of curvature in the dipoles. Equations (11) and (12) are equivalent since tuning the momentum compaction to zero implies setting the phase advance in the horizontal transverse plane. Quadrupolar perturbations due to the edge effects of the long bending magnets are implicitly computed in the phase advance functions μ_h and μ_v. Once the phases are fixed in both planes, periodic conditions are computed from the transfer matrix of the whole channel, so that conditions (6) and (7) are satisfied.
The result of minimization is very sensitive to the choice of the initial conditions. However, selecting the thin lens solution as the starting point ensures that an absolute minimum is reached for a given module. The focal distance f of each quadrupole is used in the simple relationship

$$k = \frac{1}{fl} \tag{13}$$

to give the starting values for the minimization over the focusing strength k of each quadrupole of length l. The thin lens model gives the values of the other parameters. Should the module made of thin lenses have several solutions, each set of parameters is treated separately, provided the ratio of quadrupole length to quadrupole spacing is less than unity. If not, incompatible solutions are automatically discarded. Inside the function U, the type and the number of free variables have to be chosen. The smallest set of variables contains the focusing strengths k of the quadrupoles. But, to improve the efficiency of the minimization and since the function U is purely analytic, this set can be expanded using new variables, at the expense of an increasing computing time.

3 APPLICATIONS

3.1 Periodic Structures

The isochronous period is a periodic structure achieving zero orbit dilation over one period [2]. The function (12) is to be minimized over the focusing strengths of the quadrupoles given by (13) with the following starting values for the focal distances of one cell of length L:

$$f_{1,2} = \frac{L[\pm(\cos\mu_v - \cos\mu_h) - R]}{8[-2+\cos\mu_h+\cos\mu_v]} \tag{14}$$

with

$$R = \sqrt{16[2-\cos\mu_h-\cos\mu_v]+[\cos\mu_h-\cos\mu_v]^2}. \tag{15}$$

These two focal lengths fulfill the periodicity conditions over one cell with horizontal and vertical phase advances μ_h and μ_v in the thin lens model.
Figure 1 shows an isochronous period based on 3 FODO cells of length 4 meters. The vertical phase advance is $\pi/2$, the missing magnets are in position 4 and 6, the deflection angle per bending magnet is 0.1 radian, and the quadrupoles and the bending magnets are 0.8 and 0.2 metres long.

3.2 β-Matching

A quadrupole doublet used for betatron matching is described in the following, to illustrate a case of multiple solutions, depending on the choice of the minimization function.
The aim of the doublet is to match two crossovers obeying conditions (5). In the thin lens model, the symmetrical doublet is fully determined, and the initial conditions for

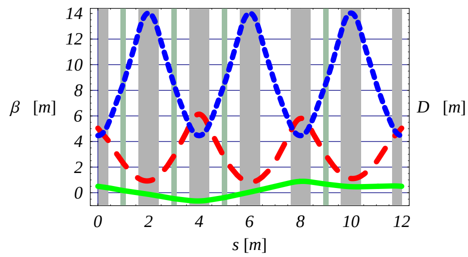

Figure 1: Horizontal (dashed line) and vertical (dotted line) β-functions and orbit dispersion (solid line) in an isochronous period.

minimization are given by the relationship between input and output betatron functions:

$$f = -\frac{\alpha_1 + \alpha_2}{\gamma_1 + \gamma_2} \quad , \quad f = \sqrt{dl}$$
$$l = \frac{\sqrt{\beta_1\gamma_2 + \beta_2\gamma_1 + 2(1 - \alpha_1\alpha_2)}}{\gamma_1 + \gamma_2} \tag{16}$$

where f is the focal length, d the spacing between the quadrupoles and l the outer drift length. Asking for $\beta_1 = \beta_2 = 1$ and $\alpha_1 = 0$, $\alpha_2 = 2$, the solution in the thin lens model is shown on figure 2.

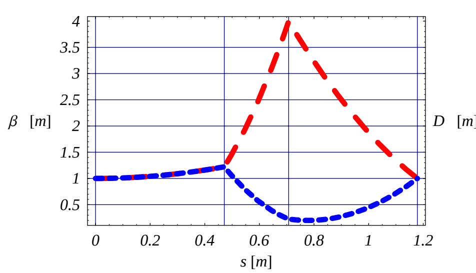

Figure 2: Horizontal (dashed line) and vertical (dotted line) β-functions for a symmetrical doublet made of thin lenses.

For the same betatron values, if $\{k, l, f\}$ is the set of variables for minimization, the function (9) leads to the solution shown on Figure 3. By construction, tiny discrepancies are expected between the two transverse planes, since the betatron functions are evaluated separately. From that point of view, the crossover is not a perfect one, but is close to that requested. On the contrary, by choosing the function

$$U = (\beta_h - \beta_v)^2 + (\alpha_h - \alpha_v)^2 \tag{17}$$

and the minimization variable $\{k\}$, a perfect crossover is reached, although with values slightly different from the requested ones (Figure 4). The pattern of the characteristic functions cannot be strictly maintained inside the module, but its variation is small.

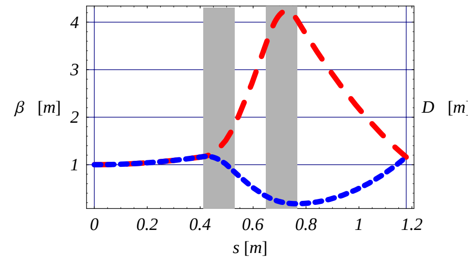

Figure 3: Horizontal (dashed line) and vertical (dotted line) β-functions for a symmetrical doublet with finite length quadrupoles of 12 cm. $\beta_h = 1.0005$ m, $\beta_v = 1.0004$ m, $\alpha_h = 2.0009$, $\alpha_v = -2.0009$.

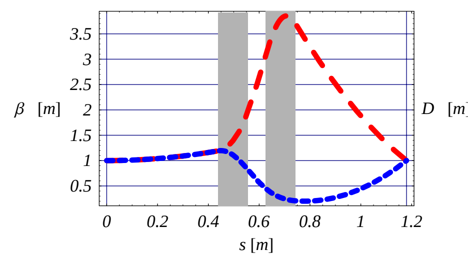

Figure 4: Horizontal (dashed line) and vertical (dotted line) β-functions for a symmetrical doublet with finite length quadrupoles of 12 cm. $\beta_h = \beta_v = 1.15955$ m, $\alpha_h = -\alpha_v = 2.34717$.

4 CONCLUSION

The method leading to the optimum optical solution takes great advantage from the analytical description of optical structures. According to the properties of each module, a specific quadratic function is built, a set of variables is chosen and numerical minimization is performed with the initial conditions given algebraically by the thin lens model. The process described leads to the automatic elaboration of complex optical modules with finite length elements and is therefore a great help in the design of new machines.

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