A PRECISE DESCRIPTION OF LONGITUDINAL MOTION WITH SYNCHROTRON RADIATION MODELLED BY "FAT PHOTONS"

Shane Koscielniak, TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C. V6T 2A3, Canada

Abstract

The equilibrium longitudinal motion of electrons emitting radiation in a synchrotron is well established, but less so the turn-by-turn dynamics. It is customary to divide the true photon emission into a smooth, continuous radiation loss and a stochastic quantum excitation with zero mean. We offer[3, 4, 5] the following improvements over previous treatments. (I) Exact equations are given for the smooth part including the exponential "saw tooth" of the reference particle, and some other second order effects. (II) The stochastic part of the per turn cumulative energy loss is *not* gaussian distributed. Using the Tchebychev-Hermite expansion about a gaussian kernel we give the *true* distribution. The technique is general and can be applied to the multiple self-convolution of any distribution function. (III) We give the effects, during the turn in which they are emitted, upon the phase advance of emitting multiple photons.

1 INTRODUCTION

The goal of this work is to derive finite difference equations (FDEs) for the longitudinal motion of an electron beam in a synchrotron. The beam will emit electromagnetic radiation and decelerate. The intention is to iterate the FDEs once per crossing of the rf cavities that accelerate the beam.

The ideal simulation of synchrotron radiation would have each emission event simulated individually. However, there are typically hundreds of photons emitted per particle per turn; and this approach is computationally prohibitive. Instead an approximate strategy is adopted. The effect is divided into a smooth, continuous radiation loss, and a stochastic photon excitation with zero mean whose cumulative effect is modelled by one 'fat photon' per turn.

1.1 FDEs in the literature

Let P_γ be the radiation power, E_γ the energy radiated per turn, E_s the synchronous particle energy, $\beta_s = v_s/c$ the kinematic parameter, J_s the longitudinal partition number, u_c the critical photon energy, and let $a \equiv (E_\gamma J_s)/(E_s \beta_s^2)$. Let p_s be synchronous momentum, η_s the slip factor, h harmonic number, $\omega_s/2\pi$ synchrotron frequency, $\hat{V}$ peak accelerating voltage and q the particle charge. Further, let us take energy and phase coordinates (ε, ϕ) relative to a sawtooth reference particle and synchronous phase Φ_s to be defined below. The following FDEs, originally stated by Bassetti and Renieri[1], have been reproduced many times:

$$\phi_{k+1} = \phi_k + 2\pi \frac{h\eta_s}{p_s v_s} \varepsilon_k \tag{1}$$

$$\varepsilon_{k+1} = \varepsilon_k (1-a) + q\hat{V}[\sin(\Phi_s + \phi_{k+1}) - \sin\Phi_s] + R \tag{2}$$

Here the stochastic part is $R = y \times 1.15026\sqrt{E_\gamma u_c}$ where y is a random variable taken from a gaussian distribution with $\bar{y} = 0$, $\bar{y^2} = 1$, $\bar{y^3} = 0$, $\bar{y^4} = 3$, etc.

The equations (1,2) are deficient in a number of respects.

- The damping term is given approximately.
- The effect upon the phase advance (during the turn in which it is emitted) of emitting damping radiation is completely omitted from the equations. To be clear, two effects are ignored: (i) that of emitting smooth, continuous radiation and (ii) that of the random emission times and values of the quanta.
- The average number of quanta emitted per particle per turn is in all existing machines too few for the Central Limit Theorem to apply; and so the distribution $P(y)$ is not gaussian.

Further, Bassetti originally omitted the average energy loss per turn. Subsequent authors included the effect, but made no attempt to transform to coordinates which remove the sawtooth energy variation.

2 DAMPING EFFECT

Under a classical description, the damping effect of radiation, along a path s in bending magnets, is given by:

$$\left[\frac{d}{ds} + \alpha_s J_s\right]\Delta p \approx -\alpha_s p_s \quad \text{where} \quad \alpha_s = \frac{P_\gamma}{v_s^2 p_s}\,. \tag{3}$$

Let ρ_s be bending radius, D dispersion, B' magnetic gradient and C machine circumference.

$$J_s = 2\left\{1 - \frac{1}{\gamma_s^2} + \frac{1}{C}\oint\left[\left.\frac{B'}{B}\right|_s + \frac{1}{2\rho(s)}\right]D(s)ds\right\}\,. \tag{4}$$

The solution of equation (3) with initial condition Δp_0 is:

$$\Delta p(s) = -p_s/J_s + [\Delta p_0 + p_s/J_s]e^{-as/\rho_s}\,. \tag{5}$$

2.1 Slip factor

The orbit period of an off-momentum particle is given by $\tau = \tau_s + \Delta\tau$ where

$$\frac{\Delta\tau}{\tau_s} = \frac{1}{C}\int_0^C \left[\frac{D(s)}{\rho(s)} - \frac{1}{\gamma_s^2}\right]\frac{\Delta p(s)}{p_s}ds \tag{6}$$

$$\approx \eta_s\left[\frac{-1}{J_s} + \left(\frac{\Delta p_0}{p_s} + \frac{1}{J_s}\right)\frac{(1-e^{-a})}{a}\right]\,. \tag{7}$$

The sawtooth reference momentum, $p_r = p_s + \Delta p_r$, is that for which the orbit-period increment $\Delta\tau = 0$, and may be found exactly[4] from (7). The synchronous phase is found by equating the energy gained in the rf cavity to the energy lost by radiation: $q\hat{V}\sin\Phi_s = E_\gamma$.

2.2 Motion relative to reference sawtooth

The motion of a general particle of momentum $p = p_s + \Delta p_r + \delta p(t)$, with respect to the sawtooth reference momentum, results in the following difference equation for momentum change per turn due to radiation loss:

$$\delta p_{k+1} = \delta p_k e^{-a} \;, \tag{8}$$

where the suffix k denotes the iteration number. The relative increment of traversal time, from (7), is given by

$$\delta\tau_k/\tau_s = \eta_s(\delta p_k/p_s)(1-e^{-a})/a \;. \tag{9}$$

This gives rise to a phase change of $\Delta\delta\phi_k = 2\pi h(\delta\tau_k/\tau_s)$.

3 QUANTUM EXCITATION

Let us assume that we are in the regime of a few hundred or more photon emissions per particle per turn. Firstly, for a single electron we must find the 'fat photon' energy loss and lumped phase advance that have equivalent effect to

- the energy decrement arising from stochastic emission of several or, possibly, many photons;
- and the cumulative phase advance arising from stochastically accumulating photon emissions.

Secondly, because a single simulation macro-particle stands in place of a very large number of electrons, we must find how to make the calculated energy decrement and phase increment representative of *many* particles.

3.1 Single-emission spectrum

Le K_ν be modified Bessel function of the second kind of fractional order $\nu = 5/3$. The probability, see Fig,1, of emitting a single photon of energy $u = xu_c$ is given by

$$u_c P_u(u) = P_X(x) = \int_x^\infty K_{5/3}(z)dz/[\Gamma(11/6)\Gamma(1/6)] \;. \tag{10}$$

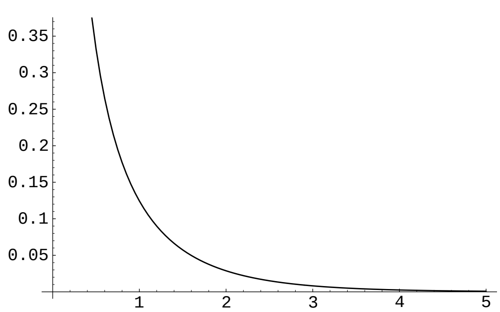

Figure 1: Single-event photon emission spectrum, P_X

3.2 Cumulative spectrum

Consider the case that an electron emits precisely n photons and let us ask what is the probability distribution of the cumulative sum $U_n \equiv \sum_{i=1}^n u_i$, where i is the event index. This cumulative n-event probability, P_{U_n}, is given by the n-fold self-convolution of P_u.

3.3 Distribution of number of photons emitted

Because the emission is a stochastic process, not all electrons emit the same number of photons. The probability to emit n photons, $\mathcal{P}_n$, follows a Poisson distribution with mean equal to m. Hence $\mathcal{P}_n \equiv e^{-m}m^n/n!$.

Based on the convolution property that means add, we have $\overline{U_n} = \int_0^\infty U \times P_{U_n}(U)dU = n\bar{U_1} = n\bar{u}$. This implies the expectation value of energy loss is given by

$$\langle\overline{U_n}\rangle = \sum_{n=0}^{\infty}\mathcal{P}_n\overline{U_n} = \bar{u}\sum_{n=0}^{\infty}\mathcal{P}_n \times n = \bar{u}\,\langle n\rangle = \bar{u}m \;. \tag{11}$$

Now, $E_\gamma = \langle\overline{U_n}\rangle$ and thus $m \equiv \langle n\rangle = E_\gamma/\bar{u}$.

3.4 Expected cumulative spectrum

To find the probability distribution of losing an energy decrement U, from any number of photon emissions, we have to form the expectation value

$$P_U \equiv \langle P_{U_n}\rangle = \sum_{n=0}^{\infty}\mathcal{P}_n P_{U_n} \approx P_{U_m} \;. \tag{12}$$

If we draw the energy decrement U of each macro-particle from the distribution $P_U(U)$ then they will be representative but still stochastic. Unfortunately, what is so conceptually simple turns out to be a formidable task when the single-event spectrum is given by the bizarre function (10).

3.5 Tchebychev-Hermite Expansion

We must find the m-fold self-convolution P_{U_m} where m may be tens, hundreds, thousands, etc.. We adopt the following strategy. First we find how the moments $\overline{U_n^k}$ evolve with n. Next we construct a function that has the same moments up to some order, and which approximates P_{U_n}.

Let $\sigma^2 \equiv \bar{u^2}$ be the variance of P_u about the mean $\bar{u}$. We define the normalized moments as: $\bar{x^k} = \bar{u^k}/\sigma^k$, with $\bar{u^k} = \int_{-\infty}^{+\infty} u^k P_u(u)du$ and introduce $X_n = U_n/\sigma$. Consider now Fourier Transforms of P_x and P_{X_n} expanded in Taylor series of the transform variable s.

$$\mathcal{FT}\{P_x\} = \sum_{k=0}^{\infty} a_k s^k \equiv A(s),\; \mathcal{FT}\{P_{X_n}\} = \sum_{k=0}^{\infty} c_k s^k \equiv C(s,n).$$

The coefficients of s^k are directly proportional to the k^{th} moments of the two distribution functions, that is

$$\overline{x^k} = [i^{-k}k!]a_k \quad \text{and} \quad \overline{X_n^k} = [i^{-k}k!]c_k(n) \;. \tag{13}$$

Suppose that we can find a suitable function $B(s) = b_0 + b_1 s + b_2 s^2 + \ldots + b_l s^l$ such that $A = e^B$ up to order s^l in A, then the evolution of the moments can be obtained from

$$\mathcal{FT}\{P_{X_n}\} = C(s) = [A(s)]^n = \exp[n \times B(s)] \;. \tag{14}$$

The coefficients b_k are called *semi-invariants*[2]. Now by a suitable choice of the coordinate origin $a_1 = 0$. Hence the first few coefficients in B are: $b_0 = b_1 = 0$, $b_2 = a_2$, $b_3 = a_3$, $b_4 = a_4 - a_2^2/2$, $b_5 = a_5 - a_2 a_3$, etc..

Evidently, as n increases, so the length scale of P_{X_n} increases as $\sqrt{n}$ or faster. Thus we introduce the variable $Z_n = X_n/\sqrt{n}$ which has a scale independent of n. We split B into a quadratic part and higher order terms. Thus

$$2\pi P_{Z_n}(z) = \int_{-\infty}^{+\infty} ds\, e^{-isz} e^{-s^2/2} \exp\left\{ n \sum_{k=3}^{\infty} \frac{s^k}{\sqrt{n^k}} b_k \right\} . \quad (15)$$

Let $f(s) = \mathcal{FT}[F(t)]$. There is the general theorem $2\pi\,(id/dz)^k\,F(t) = \int_{-\infty}^{+\infty} e^{-izs}[s^k f(s)]ds$. Hence follows the Tchebychev-Hermite[2, 3] expansion

$$P_{Z_n}(z) = \left[1 + \frac{b_3}{\sqrt{n}}\left(i\frac{d}{dz}\right)^3 + \frac{b_4}{n}\left(i\frac{d}{dz}\right)^4 + \frac{b_5}{\sqrt{n^3}}\left(i\frac{d}{dz}\right)^5 + \left(\frac{b_3^2}{n} + \frac{b_6}{n^2}\right)\left(i\frac{d}{dz}\right)^6 + \ldots\right] \frac{e^{-z^2/2}}{\sqrt{2\pi}} . \quad (16)$$

Once the derivatives are performed, and numerical values are substituted in place of the coefficients, this unwieldy form compacts into the product of a "correction polynomial" and a gaussian core. Fig.2 shows, for example, the result of trials drawn from $P_{Z_{270}}$ in the case P_u is (10).

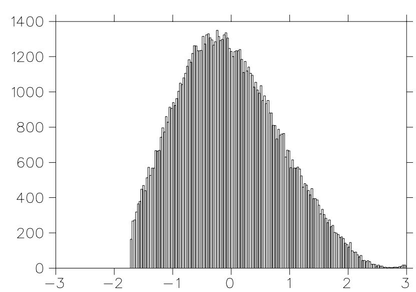

Figure 2: Histogram of P_{Z_n} for n=270 based on 10^5 trials.

3.6 Phase advance

We model the true radiation emission by the average energy loss plus a stochastically varying part with zero mean. Thus we write the single quantum energy loss as $u_i = \bar{u} + \Delta u_i$ where Δu_i is bi-polar. The energy change after m emissions becomes $U_m = m\bar{u} + \Delta U_m$ where $\Delta U_m = \sum_{i=1}^{m} \Delta u_i$. The systematic part, $m\bar{u}$ has already been included in the reference sawtooth and concerns us no longer.

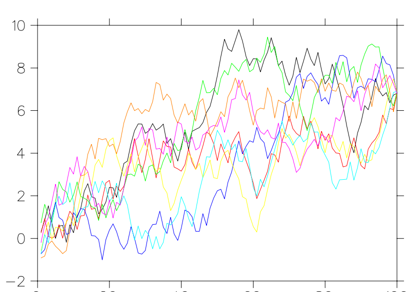

Figure 3: Examples of paths, each consisting of 100 steps, that lead to $X_m = 6.84$

We suppose the bending magnets to be isomagnetic so that the probability of photon emission is uniform. Hence we may subdivide the circumference into cells of length C/m and say that the probability of emission is unity in a cell. The average effect will be as if the photon is emitted in the centre of the cell. Thus we can treat the emission index i as the cell index and write the stochastic part:

$$\Delta\phi_{\rm stoch} = 2\pi \frac{h\eta_s}{p_s v_s} \frac{1}{m} \left[\sum_{i=1}^{m-1} \Delta U_i \;\; + \frac{1}{2}\Delta U_m \right] . \quad (17)$$

Here $\Delta U_i = \sum_{j=1}^{i} \Delta u_j$; and the individual Δu_j have zero mean. Our task is to estimate the likely value of $\Delta\phi_{\rm stoch}$ from knowledge only of ΔU_m.

We may think of the set of values ΔU_i as a path. An example set of such paths is presented in Fig.3. Of course, infinitely many paths are possible that all terminate on precisely the same value ΔU_m. For given ΔU_m, the ensemble average[5] of all possible paths, and the most likely[5] path, is $\Delta U_i = (i/m)\Delta U_m$. Hence the phase advance is

$$\langle \Delta\phi_{\rm stoch} \rangle = 2\pi(h\eta_s/p_s v_s)(\Delta U_m/2) . \quad (18)$$

4 DIFFERENCE EQUATIONS

The difference equations from the $k^{\rm th}$ to $(k+1)^{\rm th}$ turn are:

$$\phi_{k+1} = \phi_k + 2\pi \frac{h\eta_s}{p_s v_s}\left[\frac{(1-e^{-a})}{a}\varepsilon_k + \frac{1}{2}\Delta\right] \quad (19)$$

$$\varepsilon_{k+1} = \varepsilon_k e^{-a} + q\hat{V}[\sin(\Phi_s + \phi_{k+1}) - \sin\Phi_s] + \Delta . \quad (20)$$

Here $\Delta = x \times 1.00755\sqrt{E_\gamma u_c}$ is the stochastic part of the 'fat photon' and is different for each macro-particle and each turn. x is a dimensionless random variable drawn from the probability distribution of the cumulative sum of m photon emissions, in normalized coordinates. Thus $\bar{x} = 0$, $\bar{x^2} = 1$, $\bar{x^3} = 0.640018/\sqrt{m}$, $\bar{x^4} = 3 - 0.74302/m$, etc.. In the hypothetical case that $\Delta \equiv 0$ and we linearize about Φ_s, the FDEs have an exact solution: $\phi_m = \phi_0\lambda^m$, $\varepsilon_m = \varepsilon_0\lambda^m$ with $\lambda = \exp(i\mu - a/2)$ and

$$4\sin^2(\mu/2) = \omega_s^2 \frac{\sinh(a/2)}{a/2} + 2[1 - \cosh(a/2)] . \quad (21)$$

5 CONCLUSION

Iteration of the FDEs (1, 2) or (19, 20) will lead to little qualitative difference – but it is natural to wish to use exact equations (19, 20) when they are available and no extra computational cost is involved.

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